Measurement Jacobians for Multi-Camera Visual-Inertial Navigation

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Consider a 3D feature captured in the image of the base camera b with timestamp ${}^{b}t_k$. The normalized feature measurement for this is given by:

$$\mathbf{z}_k = \begin{bmatrix} \frac{x}{z} \\ \frac{y}{z} \end{bmatrix} + \mathbf{n}_k \tag{1}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = {}^{C_b({}^{b}t_k)}\mathbf{p}_f = {}^{C_b}_{I}\mathbf{R}\,{}^{I({}^{b}t_k)}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I({}^{b}t_k)}\right) + {}^{C_b}\mathbf{p}_I \tag{2}$$

Here $I({}^{b}t_k)$ refers to the state of the IMU at true imaging time ${}^{b}t_k$, which gives simple Jacobians because the clone corresponding to $I({}^{b}t_k)$ is contained in our state vector.

The chain rule gives:

$$\frac{\partial \mathbf{z}_k}{\partial \delta \mathbf{x}} = \frac{\partial \mathbf{z}_k}{\partial^{C_b({}^{b}t_k)}\delta \mathbf{p}_f}\frac{\partial^{C_b({}^{b}t_k)}\delta \mathbf{p}_f}{\partial \delta \mathbf{x}} \tag{3}$$

$$\frac{\partial \mathbf{z}_k}{\partial^{C_b({}^{b}t_k)}\delta \mathbf{p}_f} = \begin{bmatrix} \frac{1}{z} & 0 & -\frac{x}{z^2} \\ 0 & \frac{1}{z} & -\frac{y}{z^2} \end{bmatrix} \tag{4}$$

$$\frac{\partial^{C_b({}^{b}t_k)}\delta \mathbf{p}_f}{\partial^{I({}^{b}t_k)}\delta \boldsymbol{\theta}_G} = {}^{C_b}_{I}\mathbf{R}\lfloor {}^{I({}^{b}t_k)}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I({}^{b}t_k)}\right)\rfloor \tag{5}$$

$$\frac{\partial^{C_b({}^{b}t_k)}\delta \mathbf{p}_f}{\partial^{C}\delta \boldsymbol{\theta}_I} = \lfloor {}^{C_b}_{I}\mathbf{R}\,{}^{I({}^{b}t_k)}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I({}^{b}t_k)}\right)\rfloor \tag{6}$$

$$\frac{\partial^{C_b({}^{b}t_k)}\delta \mathbf{p}_f}{\partial^{G}\delta \mathbf{p}_{I({}^{b}t_k)}} = -{}^{C_b}_{I}\mathbf{R}\,{}^{I({}^{b}t_k)}_{G}\mathbf{R} \tag{7}$$

$$\frac{\partial^{C_b({}^{b}t_k)}\delta \mathbf{p}_f}{\partial^{G}\delta \mathbf{p}_f} = {}^{C_b}_{I}\mathbf{R}\,{}^{I({}^{b}t_k)}_{G}\mathbf{R} \tag{8}$$

Now let us suppose that we we capture an image from another camera, cam i, and suppose that there exists an offset in the timestamps between cam b and cam i, such that if ${}^{b}t_k = {}^{i}t_k + {}^{i}t_b$, where ${}^{b}t_k$ is the true time of the image expressed in the base camera's clock, ${}^{i}t_k$ is the reported timestamp in the measuring camera's clock, and ${}^{i}t_b$ is the unknown time offset.

For the measurement of a feature, we have:

$${}^{C_i({}^{i}t_k+{}^{i}t_b)}\mathbf{p}_f = {}^{C_i}_{I}\mathbf{R}\,{}^{I({}^{i}t_k+{}^{i}t_b)}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_f - {}^{G}\mathbf{p}_{I({}^{i}t_k+{}^{i}t_b)}\right) + {}^{C_i}\mathbf{p}_I \tag{9}$$

We do not keep an estimate for the clone associated with this time, and instead rely on interpolation. Letting ${}^{b}t_1$ and ${}^{b}t_2$ denote the bounding IMU clones between which ${}^{i}t_k + {}^{b}\hat{t}_b$ falls, we have:

$${}^{I({}^{i}t_k+{}^{i}t_b)}_{G}\mathbf{R} = \mathrm{Exp}\left(\lambda_k \mathrm{Log}\left({}^{I({}^{b}t_2)}_{G}\mathbf{R}\,{}^{G}_{I({}^{b}t_1)}\mathbf{R}\right)\right){}^{I({}^{b}t_1)}_{G}\mathbf{R} \tag{10}$$

$${}^{G}\mathbf{p}_{I({}^{i}t_k+{}^{i}t_b)} = (1-\lambda_k){}^{G}\mathbf{p}_{I({}^{b}t_1)} + \lambda_k {}^{G}\mathbf{p}_{I({}^{b}t_2)} \tag{11}$$

$$\lambda_k = \frac{{}^{i}t_k + {}^{i}t_b - {}^{b}t_1}{{}^{b}t_2 - {}^{b}t_1} \tag{12}$$

For this measurement, we can use the chain rule to compute the derivatives with respect to the bounding states and the unknown time offset:

$$\frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^{I({}^bt_1)}\delta\boldsymbol{\theta}_G} = \frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^{I({}^it_k+{}^it_b)}\delta\boldsymbol{\theta}_G}\frac{\partial^{I({}^it_k+{}^it_b)}\delta\boldsymbol{\theta}_G}{\partial^{I({}^bt_2)}\delta\boldsymbol{\theta}_G} \tag{13}$$

$$\frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^{I({}^bt_2)}\delta\boldsymbol{\theta}_G} = \frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^{I({}^it_k+{}^it_b)}\delta\boldsymbol{\theta}_G}\frac{\partial^{I({}^it_k+{}^it_b)}\delta\boldsymbol{\theta}_G}{\partial^{I({}^bt_2)}\delta\boldsymbol{\theta}_G} \tag{14}$$

$$\frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^G\delta\mathbf{p}_{I({}^bt_1)}} = \frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}\frac{\partial^G\mathbf{p}_{I({}^it_k+{}^it_b)}}{\partial^G\delta\mathbf{p}_{I({}^bt_1)}} \tag{15}$$

$$\frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^G\delta\mathbf{p}_{I({}^bt_2)}} = \frac{\delta^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}\frac{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}{\partial^G\delta\mathbf{p}_{I({}^bt_2)}} \tag{16}$$

$$\frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial\delta^it_b} = \frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}\frac{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}{\partial\delta^it_b} + \frac{\partial^{C_i({}^it_k+{}^it_b)}\delta\mathbf{p}_f}{\partial^{I({}^it_k+{}^it_b)}\delta\boldsymbol{\theta}_G}\frac{\partial^{I({}^it_k+{}^it_b)}\delta\boldsymbol{\theta}_G}{\partial\delta^it_b} \tag{17}$$

The first terms in the chain rule are identical to those derived for the cam b case, but with the calibration between the IMU to cam i. The derivatives with respect to the position are computed as:

$$\frac{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}{\partial^G\delta\mathbf{p}_{I({}^bt_1)}} = (1-\lambda_k)\mathbf{I} \tag{18}$$

$$\frac{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}{\partial^G\delta\mathbf{p}_{I({}^bt_2)}} = \lambda_k\mathbf{I} \tag{19}$$

$$\frac{\partial^G\delta\mathbf{p}_{I({}^it_k+{}^it_b)}}{\partial\delta^it_b} = \frac{1}{{}^bt_2 - {}^bt_1}\left({}^G\mathbf{p}_{I({}^bt_2)} - {}^G\mathbf{p}_{I({}^bt_1)}\right) \tag{20}$$

For the orientation derivatives, we use the following approximations for small angles $\boldsymbol{\psi}$

$$\begin{aligned}\text{Exp}\left(\boldsymbol{\theta}+\boldsymbol{\psi}\right) &\approx \text{Exp}\left(\mathbf{J}_l\left(\boldsymbol{\theta}\right)\boldsymbol{\psi}\right)\text{Exp}\left(\boldsymbol{\theta}\right) &(21)\\ &\approx \text{Exp}\left(\boldsymbol{\theta}\right)\text{Exp}\left(\mathbf{J}_r\left(\boldsymbol{\theta}\right)\boldsymbol{\psi}\right) &(22)\end{aligned}$$

where the definition of Jacobians of $SO(3)$ $\mathbf{J}_l(\cdot)$ and $\mathbf{J}_r(\cdot)$ are the following:

$$\mathbf{J}_l(\phi) = \mathbf{I} + \frac{1-cos(\parallel\phi\parallel)}{\parallel\phi\parallel^2}\lfloor\phi\times\rfloor + \frac{\parallel\phi\parallel - sin(\parallel\phi\parallel)}{\parallel\phi\parallel^3}\lfloor\phi\times\rfloor^2 \tag{23}$$

$$\mathbf{J}_r(\phi) = \mathbf{I} - \frac{1-cos(\parallel\phi\parallel)}{\parallel\phi\parallel^2}\lfloor\phi\times\rfloor + \frac{\parallel\phi\parallel - sin(\parallel\phi\parallel)}{\parallel\phi\parallel^3}\lfloor\phi\times\rfloor^2 \tag{24}$$

We also define for convenience ${}^2_1\boldsymbol{\theta} = \text{Log}\left({}^{I({}^bt_2)}_{I({}^bt_1)}\mathbf{R}\right)$. In addition we will use the fact that $\mathbf{R}\text{Exp}\left(\boldsymbol{\theta}\right) = \text{Exp}\left(\mathbf{R}\boldsymbol{\theta}\right)\mathbf{R}$. We first expand the rotation interpolation equation with respect to a perturbation in the clone $I({}^bt_1)$:

$$\mathrm{Exp}\left(-^{I(^it_k+^it_b)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^it_k+^it_b)}_{G}\mathbf{R} \tag{25}$$

$$= \mathrm{Exp}\left(\lambda_k \mathrm{Log}\left({}^{I(^bt_2)}_{G}\mathbf{R}^{G}_{I(^bt_1)}\mathbf{R}\mathrm{Exp}\left({}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\right)\right)\mathrm{Exp}\left(-^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{26}$$

$$\approx \mathrm{Exp}\left(\lambda_k\left({}^2_1\boldsymbol{\theta} + \mathbf{J}_r^{-1}\left({}^2_1\boldsymbol{\theta}\right){}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\right)\mathrm{Exp}\left(-^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{27}$$

$$\approx \mathrm{Exp}\left(\lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_r^{-1}\left({}^2_1\boldsymbol{\theta}\right){}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\mathrm{Exp}\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathrm{Exp}\left(-^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{28}$$

$$= \mathrm{Exp}\left(\lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_r^{-1}\left({}^2_1\boldsymbol{\theta}\right){}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\mathrm{Exp}\left(-\mathrm{Exp}\left(\lambda_k{}^2_1\boldsymbol{\theta}\right){}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\mathrm{Exp}\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{29}$$

$$= \mathrm{Exp}\left(\lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_r^{-1}\left({}^2_1\boldsymbol{\theta}\right){}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\mathrm{Exp}\left(-\mathrm{Exp}\left(\lambda_k{}^2_1\boldsymbol{\theta}\right){}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^it_k+^it_b)}_{G}\mathbf{R} \tag{30}$$

$$\approx \mathrm{Exp}\left(\left(\lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_r^{-1}\left({}^2_1\boldsymbol{\theta}\right) - \mathrm{Exp}\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\right){}^{I(^bt_1)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^it_k+^it_b)}_{G}\mathbf{R} \tag{31}$$

$$\Rightarrow \frac{\partial^{I(^it_k+^it_b)}\delta\boldsymbol{\theta}_G}{\partial^{I(^bt_1)}\delta\boldsymbol{\theta}_G} = -\left(\lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_r^{-1}\left({}^2_1\boldsymbol{\theta}\right) - \mathrm{Exp}\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\right) \tag{32}$$

Next we perturb $I(^bt_2)$

$$\mathrm{Exp}\left(-^{I(^it_k+^it_b)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^it_k+^it_b)}_{G}\mathbf{R} \tag{33}$$

$$= \mathrm{Exp}\left(\lambda_k \mathrm{Log}\left(\mathrm{Exp}\left(-^{I(^bt_2)}\delta\boldsymbol{\theta}_G\right)\right){}^{I(^bt_2)}_{G}\mathbf{R}^{G}_{I(^bt_1)}\mathbf{R}\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{34}$$

$$\approx \mathrm{Exp}\left(\lambda_k\left({}^2_1\boldsymbol{\theta} - \mathbf{J}_l^{-1}\left({}^2_1\boldsymbol{\theta}\right){}^{I(^bt_2)}\delta\boldsymbol{\theta}_G\right)\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{35}$$

$$\approx \mathrm{Exp}\left(-\lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_l^{-1}\left({}^2_1\boldsymbol{\theta}\right){}^{I(^bt_2)}\delta\boldsymbol{\theta}_G\right)\mathrm{Exp}\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{36}$$

$$= \mathrm{Exp}\left(-\lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_l^{-1}\left({}^2_1\boldsymbol{\theta}\right){}^{I(^bt_2)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^it_k+^it_b)}_{G}\mathbf{R} \tag{37}$$

$$\Rightarrow \frac{\partial^{I(^it_k+^it_b)}\delta\boldsymbol{\theta}_G}{\partial^{I(^bt_2)}\delta\boldsymbol{\theta}_G} = \lambda_k\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right)\mathbf{J}_l^{-1}\left({}^2_1\boldsymbol{\theta}\right) \tag{38}$$

Lastly we perturb λ_k by $\delta\lambda_k = \frac{\delta^it_b}{^bt_2 - t_b}$:

$$\mathrm{Exp}\left(-^{I(^it_k+^it_b)}\delta\boldsymbol{\theta}_G\right)\,{}^{I(^it_k+^it_b)}_{G}\mathbf{R} \tag{39}$$

$$= \mathrm{Exp}\left(\left(\lambda_k + \delta\lambda_k\right)\mathrm{Log}\left({}^{I(^bt_2)}_{G}\mathbf{R}^{G}_{I(^bt_1)}\mathbf{R}\right)\right)\,{}^{I(^bt_1)}_{G}\mathbf{R} \tag{40}$$

$$\approx \mathrm{Exp}\left(\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right){}^2_1\boldsymbol{\theta}\delta\lambda_k\right)\,{}^{I(^it_k+^it_b)}_{G}\mathbf{R} \tag{41}$$

$$\Rightarrow \frac{\partial^{I(^it_k+^it_b)}\delta\boldsymbol{\theta}_G}{\partial\delta\lambda_k} = -\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right){}^2_1\boldsymbol{\theta} \tag{42}$$

$$\Rightarrow \frac{\partial^{I(^it_k+^it_b)}\delta\boldsymbol{\theta}_G}{\partial\delta^it_b} = -\frac{1}{^bt_2 - {}^bt_1}\mathbf{J}_l\left(\lambda_k{}^2_1\boldsymbol{\theta}\right){}^2_1\boldsymbol{\theta} = -\frac{1}{^bt_2 - {}^bt_1}{}^2_1\boldsymbol{\theta} \tag{43}$$