High-Accuracy Preintegration for Visual Inertial Navigation

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1 Useful Identities

We provide some useful identities that are used in our derivations throughout the report. Given a constant angular velocity $\boldsymbol{\omega}$ between times t_1 and t_2, the rotation matrix between the two frames L_{t_1} and L_{t_2} is given by the matrix exponential:

$$\begin{aligned} {}^{L_{t_2}}_{L_{t_1}}\mathbf{R} &= \exp\left(-\lfloor \boldsymbol{\omega}(t_2 - t_1)\times\rfloor\right) \\ &= \mathbf{I}_{3\times 3} - \frac{\sin(|\boldsymbol{\omega}(t_2-t_1)|)}{|\boldsymbol{\omega}|}\lfloor\boldsymbol{\omega}\times\rfloor + \frac{1-\cos(|\boldsymbol{\omega}(t_2-t_1)|)}{|\boldsymbol{\omega}|}\lfloor\boldsymbol{\omega}\times\rfloor^2 \end{aligned} \tag{1}$$

where $\lfloor\boldsymbol{\omega}\times\rfloor = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$ The right Jacobian of $SO(3)$, $\mathbf{J}_r(\boldsymbol{\phi})$, is defined by (see [1]):

$$\mathbf{J}_r(\phi) = \mathbf{I}_{3\times 3} - \frac{1-\cos(\parallel \boldsymbol{\phi} \parallel)}{\parallel \boldsymbol{\phi} \parallel^2}\lfloor\boldsymbol{\phi}\times\rfloor + \frac{\parallel \boldsymbol{\phi} \parallel - \sin(\parallel \boldsymbol{\phi} \parallel)}{\parallel \boldsymbol{\phi} \parallel^3}\lfloor\boldsymbol{\phi}\times\rfloor^2 \tag{2}$$

Given a small angle vector perturbation $\delta\boldsymbol{\phi}$, we can make the following approximation for the rotation matrix [2] :

$$\exp\left(\lfloor\boldsymbol{\phi} + \delta\boldsymbol{\psi}\times\rfloor\right) \simeq \exp\left(\lfloor\boldsymbol{\phi}\times\rfloor\right)\exp\left(\lfloor\mathbf{J}_r(\boldsymbol{\phi})\delta\boldsymbol{\phi}\times\rfloor\right) \tag{3}$$

This allows us to map a perturbation of the Lie algebra $so(3)$ to a perturbation on the group of $SO(3)$. The JPL (natural order) quaternion is used throughout the paper [3, 4], which parametrizes the rotation (1) as follows:

$${}^{L_{t_2}}_{L_{t_1}}\bar{q} = \begin{bmatrix} \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\sin\left(\frac{|\boldsymbol{\omega}(t_2-t_1)|}{2}\right) \\ \cos\left(\frac{|\boldsymbol{\omega}(t_2-t_1)|}{2}\right) \end{bmatrix} . \tag{4}$$

2 Introduction

Visual-inertial navigation systems (VINS) that fuse visual and inertial information to provide accurate localization, have become nearly ubiquitous in part because of their low cost and light weight (e.g., see [5, 6, 7]). IMUs provide local angular velocity and linear acceleration measurements, while cameras are a cheap yet informative means for sensing the the surrounding environment and thus is an ideal aiding source for inertial navigation. In particular, these benefits have made VINS popular in resource-constrained systems such as micro aerial vehicles (MAVs) [8]. Traditionally, navigation solutions have been achieved via extended Kalman Filters (EKFs), where incoming proprioceptive (IMU) and exteroceptive (camera) measurements are processed to propagate and update state estimates, respectively. These filtering methods do not update past state estimates that have been marginalized out, thus causing them to be highly susceptible to drift due to the compounding of errors.

Graph-based optimization methods, by contrast, process all measurements taken over a trajectory simultaneously to estimate a smooth history of sensor states. These methods achieve higher accuracy due to the ability to relinearize nonlinear measurement functions and correct previous state estimates [9]. Recently, graph-based formulations have been introduced that allow the incorporation of IMU measurements into "preintegrated" factors by performing integration of the system dynamics in a *local* frame of reference [2, 10, 11]. However, these methods often simplify the required preintegrations by resorting to discrete solutions under the approximation of piece-wise constant accelerations. To improve this IMU preintegration, in this report, we instead model the IMU measurements as piece-wise constant and rigorously derive closed-form solutions of the integration equations. Based on that, we offer analytical computations of the mean, covariance, and bias Jacobians of the preintegrated measurements.

3 Analytical IMU Preintegration

An IMU typically measures the local angular velocity $\boldsymbol{\omega}$ and linear acceleration $\boldsymbol{a}$ of its body, which are assumed to be corrupted by the Gaussian white noise ($\mathbf{n}_w$ and $\mathbf{n}_a$) and the random-walk biases ($\mathbf{b}_w$ and $\mathbf{b}_a$) [4]:

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_w + \mathbf{n}_w \ , \ \mathbf{a}_m = \mathbf{a} + \mathbf{g} + \mathbf{b}_a + \mathbf{n}_a \ , \ \dot{\mathbf{b}}_w = \mathbf{n}_{wg} \ , \ \dot{\mathbf{b}}_a = \mathbf{n}_{wa} \tag{5}$$

where $\mathbf{g}$ is the gravity vector in the local frame whose global counterpart is constant (e.g., ${}^G\mathbf{g} = [0 \ \ 0 \, 9.81]^T$). The navigation state at time-step k is given by:

$$\mathbf{x}_k = \begin{bmatrix} {}^{L_k}_G\bar{q}^T & \mathbf{b}_{w_k}^T & {}^G\mathbf{v}_k^T & \mathbf{b}_{a_k}^T & {}^G\mathbf{p}_k^T \end{bmatrix}^T \tag{6}$$

where ${}^{L_k}_G\bar{q}$ is the natural order quaternion (i.e., with JPL convention [3]) that describes the rotation from frame $\{G\}$ to frame $\{L_k\}$, and ${}^G\mathbf{v}_k$ and ${}^G\mathbf{p}_k$ are the velocity and position of the k-th local frame in the global (e.g., starting) frame, respectively. The corresponding error state and $\boxplus$ operation used in batch optimization [12] can be written as (note that hereafter the transpose has been omitted for brevity):

$$\tilde{\mathbf{x}}_k = \begin{bmatrix} {}^{L_k}\delta\boldsymbol{\theta}_G & \tilde{\mathbf{b}}_{w_k} & {}^G\tilde{\mathbf{v}}_k & \tilde{\mathbf{b}}_{a_k} & {}^G\tilde{\mathbf{p}}_k \end{bmatrix} \tag{7}$$

$$\mathbf{x}_k = \hat{\mathbf{x}}_k \boxplus \tilde{\mathbf{x}}_k = \begin{bmatrix} {}^{L_k}_{\hat{L}_k}\delta\bar{q} \otimes {}^{\hat{L}_k}_G\hat{\bar{q}} \\ \hat{\mathbf{b}}_{w_k} + \tilde{\mathbf{b}}_{w_k} \\ {}^G\hat{\mathbf{v}}_k + {}^G\tilde{\mathbf{v}}_k \\ \hat{\mathbf{b}}_{a_k} + \tilde{\mathbf{b}}_{a_k} \\ {}^G\hat{\mathbf{p}}_k + {}^G\tilde{\mathbf{p}}_k \end{bmatrix} \tag{8}$$

where the operator $\otimes$ denotes quaternion multiplication, and ${}^{L_k}_{\hat{L}_k}\delta\bar{q}$ is the error quaternion whose vector portion is half the error angle, ${}^{L_k}\delta\boldsymbol{\theta}_G = 2\mathbf{vec}\left({}^{L_k}_{\hat{L}_k}\delta\bar{q}\right) := 2{}^{L_k}_{\hat{L}_k}\delta\mathbf{q}$. We here use the following unit quaternion notation $\bar{q} := \begin{bmatrix}\mathbf{q}^T & q_4\end{bmatrix}^T$. The total (full and error) states are then comprised of all these vectors.

To locally combine all the IMU measurements from time-step k to $k+1$ without accessing the state estimates (in particular, the orientation), we can perform the following factorization of the *current* rotation matrix and integration of the measurements [13]:

$$\begin{aligned} {}^G\mathbf{p}_{k+1} &= {}^G\mathbf{p}_k + {}^G\mathbf{v}_k\Delta t - \frac{1}{2}{}^G\mathbf{g}\Delta t^2 + {}^G_k\mathbf{R} \underbrace{\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} {}^k_\tau\mathbf{R}\left({}^\tau\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) d\tau ds}_{{}^k\boldsymbol{\alpha}_{k+1}} \\ &=: {}^G\mathbf{p}_k + {}^G\mathbf{v}_k\Delta t - \frac{1}{2}{}^G\mathbf{g}\Delta t^2 + {}^G_k\mathbf{R}\,{}^k\boldsymbol{\alpha}_{k+1} \end{aligned} \tag{9}$$

$$\begin{aligned} {}^G\mathbf{v}_{k+1} &= {}^G\mathbf{v}_k - {}^G\mathbf{g}\Delta t + {}^G_k\mathbf{R} \underbrace{\int_{t_k}^{t_{k+1}} {}^k_\tau\mathbf{R}\left({}^\tau\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) d\tau}_{{}^k\boldsymbol{\beta}_{k+1}} \\ &=: {}^G\mathbf{v}_k - {}^G\mathbf{g}\Delta t + {}^G_k\mathbf{R}\,{}^k\boldsymbol{\beta}_{k+1} \end{aligned} \tag{10}$$

$${}^{k+1}_G\mathbf{R} = {}^{k+1}_k\mathbf{R}\,{}^k_G\mathbf{R} \tag{11}$$

where $\Delta t = t_{k+1} - t_k$. It becomes clear that the above integrals have been collected into the *preintegrated* measurements, ${}^k\boldsymbol{\alpha}_{k+1}$ and ${}^k\boldsymbol{\beta}_{k+1}$, which are expressed in the k-th local frame. Rearrangement of these equations yields:

$$ {}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_{k} - {}^{G}\mathbf{v}_{k}\Delta t + \frac{1}{2}{}^{G}\mathbf{g}\Delta t^2\right) = {}^{k}\boldsymbol{\alpha}_{k+1}\left({}^{\tau}\mathbf{a}_m, {}^{\tau}\boldsymbol{\omega}_m, \mathbf{n}_a, \mathbf{n}_w, \mathbf{b}_a, \mathbf{b}_w\right) \tag{12} $$

$$ {}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_{k} + {}^{G}\mathbf{g}\Delta t\right) = {}^{k}\boldsymbol{\beta}_{k+1}\left({}^{\tau}\mathbf{a}_m, {}^{\tau}\boldsymbol{\omega}_m, \mathbf{n}_a, \mathbf{n}_w, \mathbf{b}_a, \mathbf{b}_w\right) \tag{13} $$

$$ {}^{k+1}_{G}\mathbf{R}{}^{k}_{G}\mathbf{R}^{\top} = {}^{k+1}_{k}\mathbf{R}\left({}^{\tau}\boldsymbol{\omega}_m, \mathbf{n}_w, \mathbf{b}_w\right)\ ,\ \forall\tau \in [t_k, t_{k+1}] \tag{14} $$

For the remainder of this paper the biases noted will refer to the those of state k, and are approximated as constant over the preintegration interval. It is important to note that in the above equations, the preintegrated measurements are explicitly expressed as the functions of the IMU measurements, noise, and true biases to show their dependency on these variables, as we will see later that it is this dependency that makes the exact preintegration impossible. To address this issue, we employ the following first-order Taylor series expansion with respect to the biases:

$$ \begin{aligned} &{}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_{k} - {}^{G}\mathbf{v}_{k}\Delta t + \frac{1}{2}{}^{G}\mathbf{g}\Delta t^2\right) \simeq \\ &\qquad {}^{k}\boldsymbol{\alpha}_{k+1}\left({}^{\tau}\mathbf{a}_m, {}^{\tau}\boldsymbol{\omega}_m, \mathbf{n}_a, \mathbf{n}_w, \bar{\mathbf{b}}_a, \bar{\mathbf{b}}_w\right) + \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{b}_a}\Big|_{\bar{\mathbf{b}}_a}\Delta\mathbf{b}_a + \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{b}_w}\Big|_{\bar{\mathbf{b}}_w}\Delta\mathbf{b}_w \end{aligned} \tag{15} $$

$$ \begin{aligned} &{}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_{k} + {}^{G}\mathbf{g}\Delta t\right) \simeq \\ &\qquad {}^{k}\boldsymbol{\beta}_{k+1}\left({}^{\tau}\mathbf{a}_m, {}^{\tau}\boldsymbol{\omega}_m, \mathbf{n}_a, \mathbf{n}_w, \bar{\mathbf{b}}_a, \bar{\mathbf{b}}_w\right) + \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{b}_a}\Big|_{\bar{\mathbf{b}}_a}\Delta\mathbf{b}_a + \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{b}_w}\Big|_{\bar{\mathbf{b}}_w}\Delta\mathbf{b}_w \end{aligned} \tag{16} $$

$$ {}^{k+1}_{G}\mathbf{R}{}^{k}_{G}\mathbf{R}^{\top} \simeq \mathbf{R}\left(\Delta\mathbf{b}_w\right){}^{k+1}_{k}\mathbf{R}\left({}^{\tau}\boldsymbol{\omega}_m, \mathbf{n}_w, \bar{\mathbf{b}}_w\right) \tag{17} $$

where the preintegration functions have been linearized about the current bias estimates, $\bar{\mathbf{b}}_w$ and $\bar{\mathbf{b}}_a$, and $\Delta\mathbf{b}_w := \mathbf{b}_w - \bar{\mathbf{b}}_w$ and $\Delta\mathbf{b}_a := \mathbf{b}_a - \bar{\mathbf{b}}_a$ are the difference between the true biases and their linearization points. Note that in the case of the relative rotations, a change in bias is modeled as inducing a further rotation on our preintegrated relative rotation. The corresponding residuals of these preintegrated measurements for use in graph optimization are given by:

$$ \delta{}^{k}\boldsymbol{\alpha}_{k+1} = {}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_{k} - {}^{G}\mathbf{v}_{k}\Delta t + \frac{1}{2}{}^{G}\mathbf{g}\Delta t^2\right) - \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{b}_a}\Delta\mathbf{b}_a - \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{b}_w}\Delta\mathbf{b}_w - {}^{k}\hat{\boldsymbol{\alpha}}_{k+1} \tag{18} $$

$$ \delta{}^{k}\boldsymbol{\beta}_{k+1} = {}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_{k} + {}^{G}\mathbf{g}\Delta t\right) - \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{b}_a}\Delta\mathbf{b}_a - \frac{\partial\boldsymbol{\beta}}{\partial\mathbf{b}_w}\Delta\mathbf{b}_w - {}^{k}\hat{\boldsymbol{\beta}}_{k+1} \tag{19} $$

$$ {}^{k+1}\delta\boldsymbol{\theta}_{k} = 2\mathbf{vec}\left({}^{k+1}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \bar{q}\left(\Delta\mathbf{b}_w\right)^{-1}\right) \tag{20} $$

where ${}^{k}\hat{\boldsymbol{\alpha}}_{k+1}$ and ${}^{k}\hat{\boldsymbol{\beta}}_{k+1}$ are the current estimated preintegrated measurements. We have defined ${}^{k+1}_{k}\breve{\bar{q}}$ as the relative quaternion found through preintegration of the IMU measurements to distinguish from a quaternion achieved by multiplication of the current *state* quaternion estimates. Based on the above definitions of ${}^{k}\boldsymbol{\alpha}_{k+1}$ and ${}^{k}\boldsymbol{\beta}_{k+1}$, we have:

$$ {}^{k}\dot{\boldsymbol{\alpha}}_{\tau} = {}^{k}\boldsymbol{\beta}_{\tau} \tag{21} $$

$$ {}^{k}\dot{\boldsymbol{\beta}}_{\tau} = {}^{k}_{\tau}\mathbf{R}\left({}^{\tau}\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) \tag{22} $$

Lastly, the rotation (quaternion) dynamics is given by:

$$ {}^{\tau}_{k}\dot{\bar{q}} = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_m - \mathbf{b}_w - \mathbf{n}_w){}^{\tau}_{k}\bar{q} \tag{23} $$

where $\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor\boldsymbol{\omega}\times\rfloor & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{\top} & 0 \end{bmatrix}$.

4 Compute Preintegration Mean and Covariance via Linear Systems

We now derive the closed-from solutions for ${}^k\hat{\boldsymbol{\alpha}}_\tau$, ${}^k\hat{\boldsymbol{\beta}}_\tau$ and ${}^\tau\hat{\bar{q}}_k$ [see (21), (22) and (23)]. In particular, the quaternion ${}^{k+1}_{k}\breve{\bar{q}}$ can be found using the zeroth order quaternion integrator [4]. In a similar fashion, we can find, in closed form, ${}^k\hat{\boldsymbol{\alpha}}_{\tau+1}$ and ${}^k\hat{\boldsymbol{\beta}}_{\tau+1}$. We begin by stacking (21) and (22), and taking the expectation to find the following linear system:

$$\begin{bmatrix} {}^k\dot{\hat{\boldsymbol{\alpha}}}_\tau \\ {}^k\dot{\hat{\boldsymbol{\beta}}}_\tau \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ {}^k_\tau\hat{\mathbf{R}} \end{bmatrix} ({}^\tau\mathbf{a}_m - \bar{\mathbf{b}}_a) \tag{24}$$

The solution of this linear dynamical system is given by:

$$\begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_{\tau+1} \\ {}^k\hat{\boldsymbol{\beta}}_{\tau+1} \end{bmatrix} = \boldsymbol{\Phi}(t_{\tau+1}, t_\tau) \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} + \int_{t_\tau}^{t_{\tau+1}} \boldsymbol{\Phi}(t_{\tau+1}, u) \begin{bmatrix} \mathbf{0} \\ {}^k_u\hat{\mathbf{R}} \end{bmatrix} ({}^u\mathbf{a}_m - \bar{\mathbf{b}}_a) du \tag{25}$$

where the state-transition matrix $\boldsymbol{\Phi}(t_{\tau+1}, t_\tau)$ is given by the matrix exponential:

$$\boldsymbol{\Phi}(t_{\tau+1}, t_\tau) = \exp\left(\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Delta t \right) = \mathbf{I} + \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Delta t + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Delta t^2 = \begin{bmatrix} \mathbf{I} & \mathbf{I}\Delta t \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{26}$$

where $\Delta t = t_{\tau+1} - t_\tau$. Substituting the state-transition into (25) yields:

$$\begin{aligned} \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_{\tau+1} \\ {}^k\hat{\boldsymbol{\beta}}_{\tau+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{I}\Delta t \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} + \int_{t_\tau}^{t_{\tau+1}} \begin{bmatrix} \mathbf{I} & \mathbf{I}(t_{\tau+1} - u) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ {}^k_u\hat{\mathbf{R}} \end{bmatrix} \hat{\mathbf{a}}\, du & (27) \\ &= \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau + {}^k\hat{\boldsymbol{\beta}}_\tau\Delta t \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} + \int_{t_\tau}^{t_{\tau+1}} \begin{bmatrix} (t_{\tau+1} - u){}^k_u\hat{\mathbf{R}}\hat{\mathbf{a}} \\ {}^k_u\hat{\mathbf{R}}\hat{\mathbf{a}} \end{bmatrix} du & (28) \\ &= \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau + {}^k\hat{\boldsymbol{\beta}}_\tau\Delta t \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} + \begin{bmatrix} {}^k_{\tau+1}\hat{\mathbf{R}} \int_{t_\tau}^{t_{\tau+1}} (t_{\tau+1} - u){}^{\tau+1}_u\hat{\mathbf{R}}\hat{\mathbf{a}}\, du \\ {}^k_{\tau+1}\hat{\mathbf{R}} \int_{t_\tau}^{t_{\tau+1}} {}^{\tau+1}_u\hat{\mathbf{R}}\hat{\mathbf{a}}\, du \end{bmatrix} & (29) \end{aligned}$$

where $\hat{\mathbf{a}} = {}^u\mathbf{a}_m - \bar{\mathbf{b}}_a$. Using $\hat{\boldsymbol{\omega}} = {}^u\boldsymbol{\omega}_m - \bar{\mathbf{b}}_w$, $\delta t = t_{\tau+1} - u$ and the matrix exponential of a skew symmetric matrix (1), as well as the Rodrigues' $SO(3)$ rotation formula, we *analytically* compute the preintegration measurement evolution from time t_τ to $t_{\tau+1}$ as follows (the detailed derivations can be found in our supplementary material):

$$\begin{aligned} \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_{\tau+1} \\ {}^k\hat{\boldsymbol{\beta}}_{\tau+1} \end{bmatrix} &= \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau + {}^k\hat{\boldsymbol{\beta}}_\tau\Delta t \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} + \begin{bmatrix} {}^k_{\tau+1}\mathbf{R} \int_0^{\Delta t} (\delta t)(\mathbf{I} - \frac{\sin(|\hat{\boldsymbol{\omega}}|(\delta t))}{|\hat{\boldsymbol{\omega}}|}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + \frac{1-\cos(|\hat{\boldsymbol{\omega}}|\delta t)}{|\hat{\boldsymbol{\omega}}|^2}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2)(\hat{\mathbf{a}})d\delta t \\ {}^k_{\tau+1}\hat{\mathbf{R}} \int_0^{\Delta t} (\mathbf{I} - \frac{\sin(|\hat{\boldsymbol{\omega}}|(\delta t))}{|\hat{\boldsymbol{\omega}}|}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + \frac{1-\cos(|\hat{\boldsymbol{\omega}}|\delta t)}{|w|^2}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2)(\hat{\mathbf{a}})d\delta t \end{bmatrix} & (30) \\ &= \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau + {}^k\hat{\boldsymbol{\beta}}_\tau\Delta t \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} \\ &\quad + \begin{bmatrix} {}^k_{\tau+1}\hat{\mathbf{R}}(\frac{(\Delta t^2)}{2}\mathbf{I} + \frac{|\hat{\boldsymbol{\omega}}|\Delta t\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|w|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t)^2 - 2\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - 2(|\hat{\boldsymbol{\omega}}|\Delta t)\sin(|\hat{\boldsymbol{\omega}}|\Delta t) + 2}{2|\hat{\boldsymbol{\omega}}|^4}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2)(\hat{\mathbf{a}}) \\ {}^k_{\tau+1}\hat{\mathbf{R}}(\Delta t\mathbf{I} - \frac{1-\cos(|\hat{\boldsymbol{\omega}}|(\Delta t))}{|\hat{\boldsymbol{\omega}}|^2}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|\hat{\boldsymbol{\omega}}|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2)(\hat{\mathbf{a}}) \end{bmatrix} & (31) \end{aligned}$$

When $\hat{\boldsymbol{\omega}}$ is very small, these equations are unstable. We therefore examine the limits as $\hat{\boldsymbol{\omega}}$ tends to zero:

$$\lim_{|\hat{\omega}|\to 0} \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_{\tau+1} \\ {}^k\hat{\boldsymbol{\beta}}_{\tau+1} \end{bmatrix} = \begin{bmatrix} {}^k\hat{\boldsymbol{\alpha}}_\tau + {}^k\hat{\boldsymbol{\beta}}_\tau\Delta t \\ {}^k\hat{\boldsymbol{\beta}}_\tau \end{bmatrix} + \begin{bmatrix} {}^k_{\tau+1}\hat{\mathbf{R}}(\frac{\Delta t^2}{2}\mathbf{I} - \frac{\Delta t^3}{3}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + \frac{\Delta t^4}{8}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2)(\hat{\mathbf{a}}) \\ {}^k_{\tau+1}\hat{\mathbf{R}}(\Delta t\mathbf{I} - \frac{\Delta t^2}{2}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + \frac{\Delta t^3}{6}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2)(\hat{\mathbf{a}}) \end{bmatrix} \tag{32}$$

5 State-Transition Matrix

In order to use these measurements, we must have the covariances associated with their error quantities. To do this, we examine the time evolution of the corresponding error states:

$$ {}^{k}\delta\dot{\boldsymbol{\alpha}}_{\tau} = {}^{k}\delta\boldsymbol{\beta}_{\tau} \tag{33}$$

$$ {}^{k}\delta\dot{\boldsymbol{\beta}}_{\tau} = {}^{k}_{\tau}\hat{\mathbf{R}}\left(\mathbf{I} + \lfloor{}^{\tau}\delta\boldsymbol{\theta}_{k}\times\rfloor\right)\left({}^{\tau}\mathbf{a}_{m} - \mathbf{b}_{a} - \mathbf{n}_{a}\right) - {}^{k}_{\tau}\hat{\mathbf{R}}\left({}^{\tau}\mathbf{a}_{m} - \hat{\mathbf{b}}_{a}\right) \tag{34}$$

$$ = {}^{k}_{\tau}\hat{\mathbf{R}}\left(-\Delta\mathbf{b}_{a} - \mathbf{n}_{a}\right) + {}^{k}_{\tau}\hat{\mathbf{R}}\lfloor{}^{\tau}\delta\boldsymbol{\theta}_{k}\times\rfloor\left({}^{\tau}\mathbf{a}_{m} - \hat{\mathbf{b}}_{a}\right) \tag{35}$$

$$ {}^{\tau}\delta\dot{\boldsymbol{\theta}}_{k} = -\left\lfloor\left(\hat{\boldsymbol{\omega}} - \hat{\mathbf{b}}_{w}\right)\times\right\rfloor{}^{\tau}\delta\boldsymbol{\theta}_{k} - \Delta\mathbf{b}_{w} - \mathbf{n}_{w} \tag{36}$$

where we have used the standard error associated with JPL-convention quaternions, ${}^{\tau}\bar{q}_{k} = \delta\bar{q}\otimes{}^{\tau}\hat{\bar{q}}_{k}$, and $\delta\bar{q} \simeq \begin{bmatrix}\frac{\delta\boldsymbol{\theta}}{2}\\ 1\end{bmatrix}$. This yields the following linearized system describing our error states:

$$\begin{bmatrix}{}^{k}\delta\dot{\boldsymbol{\alpha}}_{\tau}\\ {}^{k}\delta\dot{\boldsymbol{\beta}}_{\tau}\\ {}^{\tau}\delta\dot{\boldsymbol{\theta}}_{k}\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -{}^{k}_{\tau}\hat{\mathbf{R}}\lfloor\left({}^{\tau}\mathbf{a}_{m} - \bar{\mathbf{b}}_{a}\right)\times\rfloor\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\lfloor\left({}^{\tau}\boldsymbol{\omega}_{m} - \bar{\mathbf{b}}_{w}\right)\times\rfloor\end{bmatrix}\begin{bmatrix}{}^{k}\delta\boldsymbol{\alpha}_{\tau}\\ {}^{k}\delta\boldsymbol{\beta}_{\tau}\\ {}^{\tau}\delta\boldsymbol{\theta}_{k}\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3}\\ -{}^{k}_{\tau}\hat{\mathbf{R}} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & -\mathbf{I}_{3\times3}\end{bmatrix}\begin{bmatrix}\mathbf{n}_{a}\\ \mathbf{n}_{w}\end{bmatrix} \tag{37}$$

$$\Rightarrow\quad \dot{\mathbf{r}} = \mathbf{F}\mathbf{r} + \mathbf{G}\mathbf{n} \tag{38}$$

We ignore the error terms associated with bias, as the biases are being evaluated at their current estimates, essentially linearizing the measurements and their corresponding errors about those points. Under the approximation that $\mathbf{F}$ (38) does not change over a measurement time interval $[t_{\tau}, t_{\tau+1}]$, the discrete-time state transition matrix, can be written as:

$$\boldsymbol{\Phi}(t_{\tau+1}, t_{\tau}) = \exp(\mathbf{F}(\Delta t)) = \mathbf{I}_{3\times3} + \mathbf{F}\Delta t + \frac{1}{2}(\mathbf{F}\Delta t)^{2} + ... \tag{39}$$

where $\Delta t = t_{k+1} - t_{k}$. In order to find the analytical expression, we need to look at the powers of $\mathbf{F}$.

$$\mathbf{F} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor\end{bmatrix},\ \text{with } \mathbf{E} = -{}^{\mathrm{k}}_{\tau}\hat{\mathbf{R}}\left\lfloor\left(\mathbf{a}_{\mathrm{m}} - \hat{\mathbf{b}}_{\mathrm{a}}\right)\times\right\rfloor \text{ and } \mathbf{d} = -(\boldsymbol{\omega}_{\mathrm{m}} - \hat{\mathbf{b}}_{\mathrm{w}}) \tag{40}$$

Based on this, we have:

$$\mathbf{F}^{2} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor\end{bmatrix}\begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor^{2}\end{bmatrix} \tag{41}$$

$$\mathbf{F}^{3} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor^{2}\end{bmatrix}\begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor^{2}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor^{3}\end{bmatrix} \tag{42}$$

$$\mathbf{F}^{4} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor^{2}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor^{3}\end{bmatrix}\begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor^{2}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{E}\lfloor\mathbf{d}\times\rfloor^{3}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \lfloor\mathbf{d}\times\rfloor^{4}\end{bmatrix} \tag{43}$$

$$\mathbf{F}^{5} = \cdots$$

By close inspection, it is not difficult to see that the discrete-time state transition matrix takes the form [14]:

$$\boldsymbol{\Phi}(t_{\tau+1}, t_{\tau}) = \begin{bmatrix}\boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13}\\ \mathbf{0}_{3\times3} & \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23}\\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{\Phi}_{33}\end{bmatrix} \tag{44}$$

Based on (40)-(43), we find each entry of the state transition matrix as follows:

$$\mathbf{\Phi}_{11} = \mathbf{\Phi}_{22} = \mathbf{I}_{3\times3} \tag{45}$$

$$\mathbf{\Phi}_{12} = \mathbf{I}_{3\times3}\Delta t \tag{46}$$

$$\mathbf{\Phi}_{13} = \frac{1}{2}\mathbf{E}(\Delta t)^2 + \frac{1}{3!}\mathbf{E}\lfloor\mathbf{d}\times\rfloor(\Delta t)^3 + \frac{1}{4!}\mathbf{E}\lfloor\mathbf{d}\times\rfloor^2(\Delta t)^4 + \cdots \tag{47}$$

$$\mathbf{\Phi}_{21} = \mathbf{\Phi}_{31} = \mathbf{\Phi}_{32} = \mathbf{0}_{3\times3} \tag{48}$$

$$\mathbf{\Phi}_{23} = \mathbf{E}\Delta t + \frac{1}{2}\mathbf{E}\lfloor\mathbf{d}\times\rfloor\Delta t^2 + \frac{1}{3!}\mathbf{E}\lfloor\mathbf{d}\times\rfloor^2\Delta t^3 + \frac{1}{4!}\mathbf{E}\lfloor\mathbf{d}\times\rfloor^3\Delta t^4 + \cdots \tag{49}$$

$$\mathbf{\Phi}_{33} = \mathbf{I}_{3\times3} + \lfloor\mathbf{d}\times\rfloor\Delta t + \lfloor\mathbf{d}\times\rfloor^2\Delta t^2 + \cdots \tag{50}$$

We now first turn our attention to $\mathbf{\Phi}_{13}$. Using $\lfloor\mathbf{d}\times\rfloor^3 = -|\mathbf{d}|^2\lfloor\mathbf{d}\times\rfloor$, we can write it as:

$$\begin{aligned}
\mathbf{\Phi}_{13} &= \tfrac{1}{2}\mathbf{E}(\Delta t)^2 + \tfrac{1}{3!}\mathbf{E}\lfloor\mathbf{d}\times\rfloor(\Delta t)^3 + \tfrac{1}{4!}\mathbf{E}\lfloor\mathbf{d}\times\rfloor^2(\Delta t)^4 - \tfrac{1}{5!}|\mathbf{d}|^2\mathbf{E}\lfloor\mathbf{d}\times\rfloor(\Delta t)^5 - \tfrac{1}{6!}|\mathbf{d}|^2\mathbf{E}\lfloor\mathbf{d}\times\rfloor^2(\Delta t)^6 + \tfrac{1}{7!}|\mathbf{d}|^4\mathbf{E}\lfloor\mathbf{d}\times\rfloor(\Delta t)^7 + \ldots \\
&= \frac{1}{2}\mathbf{E}(\Delta t)^2 + \mathbf{E}\left(\frac{1}{3!}(\Delta t)^3 - \frac{1}{5!}|\mathbf{d}|^2(\Delta t)^5 + \ldots\right)\lfloor\mathbf{d}\times\rfloor + \mathbf{E}\left(\frac{1}{4!}(\Delta t)^4 - \frac{1}{6!}|\mathbf{d}|^2(\Delta t)^6 + \ldots\right)\lfloor\mathbf{d}\times\rfloor^2 \\
&= \frac{1}{2}\mathbf{E}(\Delta t)^2 + \frac{\mathbf{E}}{|\mathbf{d}|^3}(|\mathbf{d}|\Delta t - \sin(|\mathbf{d}|\Delta t))\lfloor\mathbf{d}\times\rfloor + \frac{\mathbf{E}}{|\mathbf{d}|^4}(\cos(|\mathbf{d}|\Delta t) - 1 + \frac{1}{2}(|\mathbf{d}|\Delta t)^2)\lfloor\mathbf{d}\times\rfloor^2
\end{aligned} \tag{51}$$

Similarly, we repeat this process for $\mathbf{\Phi}_{23}$:

$$\begin{aligned}
\mathbf{\Phi}_{23} &= \mathbf{E}\Delta t + \frac{1}{2}\mathbf{E}\lfloor\mathbf{d}\times\rfloor\Delta t^2 + \frac{1}{3!}\mathbf{E}\lfloor\mathbf{d}\times\rfloor^2\Delta t^3 - \frac{1}{4!}|\mathbf{d}|^2\mathbf{E}\lfloor\mathbf{d}\times\rfloor\Delta t^4 + \ldots \\
&= \mathbf{E}\Delta t + \mathbf{E}\left(\frac{1}{2}\Delta t^2 - \frac{1}{4!}|\mathbf{d}|^2\Delta t^4 + \ldots\right)\lfloor\mathbf{d}\times\rfloor + \mathbf{E}\left(\frac{1}{3!}\Delta t^3 - \frac{1}{5!}|\mathbf{d}|^2\Delta t^5 + \ldots\right)\lfloor\mathbf{d}\times\rfloor^2 \\
&= \mathbf{E}\Delta t + \frac{\mathbf{E}}{|\mathbf{d}|^2}(1 - \cos(|\mathbf{d}|\Delta t))\lfloor\mathbf{d}\times\rfloor + \frac{\mathbf{E}}{|\mathbf{d}|^3}(|\mathbf{d}|\Delta t - \sin(|\mathbf{d}|\Delta t))\lfloor\mathbf{d}\times\rfloor^2
\end{aligned} \tag{52}$$

Lastly, $\mathbf{\Phi}_{33}$ is simply computed based on the Rodriques' formula (1):

$$\mathbf{\Phi}_{33} = \exp(\lfloor\mathbf{d}\times\rfloor\Delta t) = \mathbf{I}_{3\times3} + \frac{\sin(|\mathbf{d}|\Delta t)}{|\mathbf{d}|}\lfloor\mathbf{d}\times\rfloor + \frac{1 - \cos(|\mathbf{d}|\Delta t)}{|\mathbf{d}|^2}\lfloor\mathbf{d}\times\rfloor^2 \tag{53}$$

When ω, similarly when $\mathbf{d}$, becomes small the above equations will become numerically unstable due to $|\mathbf{d}|$, and thus ω, appearing in the denominator. We therefore look to take the limit as $|\mathbf{d}|$ approaches zero.

$$\lim_{|\mathbf{d}|\to0}\mathbf{\Phi}_{13} = \mathbf{E}\left(\frac{1}{2}\mathbf{I}_{3\times3}(\Delta t)^2 + \frac{\Delta t^3}{6}\lfloor\mathbf{d}\times\rfloor + \frac{\Delta t^4}{24}\lfloor\mathbf{d}\times\rfloor^2\right) \tag{54}$$

$$\lim_{|\mathbf{d}|\to0}\mathbf{\Phi}_{23} = \mathbf{E}\left(\Delta t\mathbf{I}_{3\times3} + \frac{\Delta t^2}{2}\lfloor\mathbf{d}\times\rfloor + \frac{\Delta t^3}{6}\lfloor\mathbf{d}\times\rfloor^2\right) \tag{55}$$

$$\lim_{|\mathbf{d}|\to0}\mathbf{\Phi}_{33} = \mathbf{I}_{3\times3} + \Delta t\lfloor\mathbf{d}\times\rfloor + \frac{\Delta t^2}{2}\lfloor\mathbf{d}\times\rfloor^2 \tag{56}$$

6 Discrete Covariance Propagation

Using the expressions for the state-transition matrix, the covariance propagation for our preintegrated measurements takes the form:

$$\mathbf{P}_{t_k} = \mathbf{0}_{9\times 9} \tag{57}$$

$$\mathbf{P}_{t_{\tau+1}} = \boldsymbol{\Phi}(t_{\tau+1}, t_\tau)\mathbf{P}_{t_\tau}\boldsymbol{\Phi}(t_{\tau+1}, t_\tau)^\top + \mathbf{Q}_d \tag{58}$$

$$\mathbf{Q}_d = \int_{t_\tau}^{t_{\tau+1}} \boldsymbol{\Phi}(t_{\tau+1}, u)\mathbf{G}(u)\mathbf{Q}_c\mathbf{G}^\top(u)\boldsymbol{\Phi}^\top(t_{\tau+1}, u)du \tag{59}$$

$$= \int_{t_\tau}^{t_{\tau+1}} \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13} \\ \mathbf{0}_{3\times3} & \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{\Phi}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ -{}^k_u\hat{\mathbf{R}} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & -\mathbf{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \sigma_a^2\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \sigma_w^2\mathbf{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{3\times3} & -{}^u_k\hat{\mathbf{R}} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\mathbf{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{11}^\top & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \boldsymbol{\Phi}_{12}^\top & \boldsymbol{\Phi}_{22}^\top & \mathbf{0}_{3\times3} \\ \boldsymbol{\Phi}_{13}^\top & \boldsymbol{\Phi}_{23}^\top & \boldsymbol{\Phi}_{33}^\top \end{bmatrix} du$$

$$= \int_{t_\tau}^{t_{\tau+1}} \begin{bmatrix} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13} \\ \mathbf{0}_{3\times3} & \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{\Phi}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \sigma_a^2\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \sigma_w^2\mathbf{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{11}^\top & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \boldsymbol{\Phi}_{12}^\top & \boldsymbol{\Phi}_{22}^\top & \mathbf{0}_{3\times3} \\ \boldsymbol{\Phi}_{13}^\top & \boldsymbol{\Phi}_{23}^\top & \boldsymbol{\Phi}_{33}^\top \end{bmatrix} du$$

$$= \int_{t_\tau}^{t_{\tau+1}} \begin{bmatrix} \sigma_a^2\boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{12}^\top + \sigma_w^2\boldsymbol{\Phi}_{13}\boldsymbol{\Phi}_{13}^\top & \sigma_a^2\boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{22}^\top + \sigma_w^2\boldsymbol{\Phi}_{13}\boldsymbol{\Phi}_{23}^\top & \sigma_w^2\boldsymbol{\Phi}_{13}\boldsymbol{\Phi}_{33}^\top \\ (\sigma_a^2\boldsymbol{\Phi}_{12}\boldsymbol{\Phi}_{22}^\top + \sigma_w^2\boldsymbol{\Phi}_{13}\boldsymbol{\Phi}_{23}^\top)^\top & \sigma_a^2\boldsymbol{\Phi}_{22}\boldsymbol{\Phi}_{22}^\top + \sigma_w^2\boldsymbol{\Phi}_{23}\boldsymbol{\Phi}_{23}^\top & \sigma_w^2\boldsymbol{\Phi}_{23}\boldsymbol{\Phi}_{33}^\top \\ (\sigma_w^2\boldsymbol{\Phi}_{13}\boldsymbol{\Phi}_{33}^\top)^\top & (\sigma_w^2\boldsymbol{\Phi}_{23}\boldsymbol{\Phi}_{33}^\top)^\top & \sigma_w^2\boldsymbol{\Phi}_{33}\boldsymbol{\Phi}_{33}^\top \end{bmatrix} du \tag{60}$$

Defining $\delta t = t_{k+1} - u$, each of these integration entries can be written as $\int_{t_\tau}^{t_{\tau+1}} \mathbf{f}(t_{\tau+1} - u)du = \int_0^{\Delta t} \mathbf{f}(\delta t)d\delta t$. Using the expressions for the state-transition matrix, the discrete time noise covariance can be found:

$$\mathbf{Q}_{11} = \sigma_a^2\mathbf{I}_{3\times3}\frac{\Delta t^3}{3} + \sigma_w^2\mathbf{E}\left(\frac{\Delta t^5}{20}\mathbf{I}_{3\times3} - \frac{1}{60|\mathbf{d}|^7}\Big(120(|\mathbf{d}|\Delta t) - 240\text{sin}(|\mathbf{d}|\Delta t) + 20(|\mathbf{d}|\Delta t)^3 - 3(|\mathbf{d}|\Delta t)^5 + 120(|\mathbf{d}|\Delta t)\text{cos}(|\mathbf{d}|\Delta t)\Big)\lfloor\mathbf{d}\times\rfloor^2\right)\mathbf{E}^\top$$

$$\mathbf{Q}_{12} = \sigma_a^2\frac{\Delta t^2}{2}\mathbf{I}_{3\times3} + \sigma_w^2\mathbf{E}\left(\frac{\Delta t^4}{8}\mathbf{I}_{3\times3} + \frac{3\text{sin}(|\mathbf{d}|\Delta t) - 2(|\mathbf{d}|\Delta t) - (|\mathbf{d}|\Delta t)\text{cos}(|\mathbf{d}|\Delta t)}{|\mathbf{d}|^5}\lfloor\mathbf{d}\times\rfloor + \frac{8\text{cos}(|\mathbf{d}|\Delta t) - 4(|\mathbf{d}|\Delta t)^2 + (|\mathbf{d}|\Delta t)^4 + 8(|\mathbf{d}|\Delta t)\text{sin}(|\mathbf{d}|\Delta t) - 8}{8|\mathbf{d}|^6}\lfloor\mathbf{d}\times\rfloor^2\right)\mathbf{E}^\top$$

$$\mathbf{Q}_{13} = \sigma_w^2\mathbf{E}\left(\mathbf{I}_{3\times3}\frac{1}{6}\Delta t^3 + \frac{2\text{cos}(|\mathbf{d}|\Delta t) + |\mathbf{d}|\Delta t\text{sin}(|\mathbf{d}|\Delta t) - 2}{|\mathbf{d}|^4}\lfloor\mathbf{d}\times\rfloor + \frac{6|\mathbf{d}|\Delta t - 12\text{sin}(|\mathbf{d}|\Delta t) + (|\mathbf{d}|\Delta t)^3 + 6|\mathbf{d}|\Delta t\text{cos}(|\mathbf{d}|\Delta t)}{6|\mathbf{d}|^5}\lfloor\mathbf{d}\times\rfloor^2\right)$$

$$\mathbf{Q}_{22} = \sigma_a^2\Delta t\mathbf{I}_{3\times3} + \sigma_w^2\mathbf{E}\left(\frac{\Delta t^3}{3}\mathbf{I}_{3\times3} + \frac{6\text{sin}(|\mathbf{d}|\Delta t) - 6(|\mathbf{d}|\Delta t) + (|\mathbf{d}|\Delta t)^3}{3|\mathbf{d}|^5}\lfloor\mathbf{d}\times\rfloor^2\right)\mathbf{E}^\top$$

$$\mathbf{Q}_{23} = \sigma_w^2\mathbf{E}\left(\frac{\Delta t^2}{2}\mathbf{I}_{3\times3} + \frac{\text{sin}(|\mathbf{d}|\Delta t) - (|\mathbf{d}|\Delta t)}{|\mathbf{d}|^3}\lfloor\mathbf{d}\times\rfloor - \frac{4\text{sin}^2(\frac{|\mathbf{d}|\Delta t}{2}) - (|\mathbf{d}|\Delta t)^2}{2|\mathbf{d}|^4}\lfloor\mathbf{d}\times\rfloor^2\right)$$

$$\mathbf{Q}_{33} = \Delta t\sigma_w^2\mathbf{I}_{3\times3}$$

$$\mathbf{Q}_{21} = \mathbf{Q}_{12}^\top$$

$$\mathbf{Q}_{31} = \mathbf{Q}_{13}^\top$$

$$\mathbf{Q}_{32} = \mathbf{Q}_{23}^\top$$

When ω, similarly when $\mathbf{d}$, becomes small the above equations will become numerically unstable due to

$|\mathbf{d}|$, and thus ω, appearing in the denominator. We therefore look to take the limit as $|\mathbf{d}|$ approaches zero.

$$\lim_{|\mathbf{d}|\to 0} \mathbf{Q}_{11} = \sigma_a^2 \mathbf{I}\frac{\Delta t^3}{3} + \sigma_w^2 \mathbf{E}(\frac{\Delta t^5}{20}\mathbf{I} + \frac{\Delta t^7}{504}\lfloor \mathbf{d}\times\rfloor^2)\mathbf{E}^\top \tag{61}$$

$$\lim_{|\mathbf{d}|\to 0} \mathbf{Q}_{12} = \sigma_a^2 \frac{\Delta t^2}{2}\mathbf{I} + \sigma_w^2 \mathbf{E}(\frac{\Delta t^4}{8}\mathbf{I} - \frac{\Delta t^5}{60}\lfloor \mathbf{d}\times\rfloor + \frac{\Delta t^6}{144}\lfloor \mathbf{d}\times\rfloor^2)\mathbf{E}^\top \tag{62}$$

$$\lim_{|\mathbf{d}|\to 0} \mathbf{Q}_{13} = \sigma_w^2 \mathbf{E}(\mathbf{I}\frac{1}{6}\Delta t^3 - \frac{\Delta t^4}{12}\lfloor \mathbf{d}\times\rfloor + \frac{\Delta t^5}{40}\lfloor \mathbf{d}\times\rfloor^2) \tag{63}$$

$$\lim_{|\mathbf{d}|\to 0} \mathbf{Q}_{22} = \sigma_a^2 \Delta t \mathbf{I} + \sigma_w^2 \mathbf{E}(\frac{\Delta t^3}{3}\mathbf{I} + \frac{\Delta t^5}{60}\lfloor \mathbf{d}\times\rfloor^2)\mathbf{E}^\top \tag{64}$$

$$\lim_{|\mathbf{d}|\to 0} \mathbf{Q}_{23} = \sigma_w^2 \mathbf{E}(\frac{\mathbf{I}\Delta t^2}{2} - \frac{\Delta t^3}{6}\lfloor \mathbf{d}\times\rfloor + \frac{\Delta t^4}{24}\lfloor \mathbf{d}\times\rfloor^2) \tag{65}$$

$$\lim_{|\mathbf{d}|\to 0} \mathbf{Q}_{33} = \Delta t \sigma_w^2 \mathbf{I} \tag{66}$$

$$\tag{67}$$

7 Bias Jacobians

Changes in biases are modeled as adding corrections to our preintegration measurements through the use of bias Jacobians [see (15) and (16)], it is critical to compute these Jacobians accurately. In particular, as seen from (31) that each update term is linear in the estimated acceleration, $\hat{\mathbf{a}} = \mathbf{a}_m - \bar{\mathbf{b}}_a$, we can find the bias Jacobians of ${}^k\boldsymbol{\alpha}_{k+1}$ and ${}^k\boldsymbol{\beta}_{k+1}$ with respect to $\mathbf{b}_a$ as follows:

$$\begin{bmatrix} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_a} \\ \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_a} \end{bmatrix} =: \begin{bmatrix} \mathbf{H}_\alpha(\tau+1) \\ \mathbf{H}_\beta(\tau+1) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_\alpha(\tau) + \mathbf{H}_\beta(\tau)\Delta t \\ \mathbf{H}_\beta(\tau) \end{bmatrix} \tag{68}$$

$$- \begin{bmatrix} {}^k_{\tau+1}\mathbf{R}\left(\frac{\Delta t^2}{2}\mathbf{I}_{3\times 3} + \frac{|w|\Delta t \cos(|w|\Delta t) - \sin(|w|\Delta t)}{|w|^3}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor + \frac{(|w|\Delta t)^2 - 2\cos(|w|\Delta t) - 2(|w|\Delta t)\sin(|w|\Delta t) + 2}{2|w|^4}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor^2\right) \\ {}^k_{\tau+1}\mathbf{R}\left(\Delta t \mathbf{I}_{3\times 3} - \frac{1-\cos(|\boldsymbol{\omega}|(\Delta t))}{|w|^2}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor + \frac{(|w|\Delta t) - \sin(|\boldsymbol{\omega}|\Delta t)}{|w|^3}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor^2\right) \end{bmatrix}$$

$$\lim_{|w|\to 0} \begin{bmatrix} \mathbf{H}_\alpha(\tau+1) \\ \mathbf{H}_\beta(\tau+1) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_\alpha(\tau) + \mathbf{H}_\beta(\tau)\Delta t \\ \mathbf{H}_\beta(\tau) \end{bmatrix} - \begin{bmatrix} {}^k_{\tau+1}\mathbf{R}(\frac{\Delta t^2}{2}\mathbf{I} - \frac{\Delta t^3}{3}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor + \frac{\Delta t^4}{8}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor^2) \\ {}^k_{\tau+1}\mathbf{R}(\Delta t \mathbf{I} - \frac{\Delta t^2}{2}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor + \frac{\Delta t^3}{6}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor^2) \end{bmatrix} \tag{69}$$

Similarly, the Jacobians with respect to the gyro bias $\frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_w} =: \mathbf{J}_\alpha$, $\frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_w} =: \mathbf{J}_\beta$ can be found by taking the derivative with respect to each of gyro bias entries. That is, we seek the entries of $\mathbf{J}_\alpha = \begin{bmatrix} \frac{\partial^k \boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{w_1}} & \frac{\partial^k \boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{w_2}} & \frac{\partial^k \boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{w_3}} \end{bmatrix}$ and $\mathbf{J}_\beta = \begin{bmatrix} \frac{\partial^k \boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{w_1}} & \frac{\partial^k \boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{w_2}} & \frac{\partial^k \boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{w_3}} \end{bmatrix}$. For each entry we derive the following:

$$\begin{aligned} \frac{\partial^k \boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{w_i}} &= \frac{\partial^k \boldsymbol{\alpha}_{\tau}}{\partial \mathbf{b}_{w_i}} + \frac{\partial^k \boldsymbol{\beta}_{\tau}\Delta t}{\partial \mathbf{b}_{w_i}} \\ &\quad + \frac{\partial}{\partial \mathbf{b}_{w_i}}\Bigg({}^k_{\tau+1}\mathbf{R}(\frac{(\Delta t^2)}{2}\mathbf{I} + \frac{|\hat{\boldsymbol{\omega}}|\Delta t \cos(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|\hat{\boldsymbol{\omega}}|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor \\ &\quad + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t)^2 - 2\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - 2(|\hat{\boldsymbol{\omega}}|\Delta t)\sin(|\hat{\boldsymbol{\omega}}|\Delta t) + 2}{2|\hat{\boldsymbol{\omega}}|^4}\lfloor \hat{\boldsymbol{\omega}}\times\rfloor^2)\Bigg)(\hat{\mathbf{a}}) \end{aligned} \tag{70}$$

The first two terms can be found with the previous derivatives. Defining $\hat{\mathbf{e}}_i$ as the unit vector in the ith direction, the third term can be found as:

$$\frac{\partial^k_{\tau+1}\mathbf{R}(\frac{(\Delta t)^2}{2}\mathbf{I} + f_1\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + f_2\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2)}{\partial \mathbf{b}_{w_i}} = \frac{\partial^k_{\tau+1}\mathbf{R}}{\partial \mathbf{b}_{w_i}}\left(\frac{(\Delta t)^2}{2}\mathbf{I} + f_1\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + f_2\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2\right)$$
$$+ {}^k_{\tau+1}\mathbf{R}\left(\frac{\partial f_1}{\partial \mathbf{b}_{w_i}}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor - f_1\lfloor\hat{\mathbf{e}}_i\times\rfloor + \frac{\partial f_2}{\partial \mathbf{b}_{w_i}}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2 - f_2(\lfloor\hat{\mathbf{e}}_i\times\rfloor\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + \lfloor\hat{\boldsymbol{\omega}}\times\rfloor\lfloor\hat{\mathbf{e}}_i\times\rfloor)\right) \quad (71)$$

where f_1 and f_2 are the corresponding coefficients in (70), and their derivatives are computed as:

$$\frac{\partial f_1}{\partial \mathbf{b}_{w_i}} = \frac{\boldsymbol{\omega}^\top\hat{\mathbf{e}}_i(|\hat{\boldsymbol{\omega}}|^2\Delta t^2\text{sin}(|\hat{\boldsymbol{\omega}}|\Delta t) - 3\text{sin}(|\hat{\boldsymbol{\omega}}|\Delta t) + 3|\hat{\boldsymbol{\omega}}|\Delta t\text{cos}(|\hat{\boldsymbol{\omega}}|\Delta t))}{|\hat{\boldsymbol{\omega}}|^5} \quad (72)$$
$$\frac{\partial f_2}{\partial \mathbf{b}_{w_i}} = \frac{\boldsymbol{\omega}^\top\hat{\mathbf{e}}_i((|\hat{\boldsymbol{\omega}}|\Delta t)^2 - 4\text{cos}(|\hat{\boldsymbol{\omega}}|\Delta t) - 4(|\hat{\boldsymbol{\omega}}|\Delta t)\text{sin}(|\hat{\boldsymbol{\omega}}|\Delta t) + (|\hat{\boldsymbol{\omega}}|\Delta t)^2\text{cos}(|\hat{\boldsymbol{\omega}}|\Delta t) + 4)}{|\hat{\boldsymbol{\omega}}|^6} \quad (73)$$

For small $\hat{\boldsymbol{\omega}}$,

$$\lim_{|w|\to 0}\frac{\partial f_1}{\partial \mathbf{b}_{w_i}} = -\hat{\boldsymbol{\omega}}_i\frac{\Delta t^5}{15} \quad (74)$$

$$\lim_{|w|\to 0}\frac{\partial f_2}{\partial \mathbf{b}_{w_i}} = \hat{\boldsymbol{\omega}}_i\frac{\Delta t^6}{72} \quad (75)$$

Similarly, we have (see (31)):

$$\frac{\partial^k\boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{w_i}} = \frac{\partial^k\boldsymbol{\beta}_{\tau}}{\partial \mathbf{b}_{w_i}} + \frac{\partial^k_{\tau+1}\mathbf{R}}{\partial \mathbf{b}_{w_i}}\left(\Delta t\mathbf{I} + f_3\lfloor\hat{\boldsymbol{\omega}}\times\rfloor + f_4\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2\right)$$
$$+ {}^k_{\tau+1}\hat{\mathbf{R}}\left(\frac{\partial f_3}{\partial \mathbf{b}_{w_i}}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor - f_3\lfloor\hat{\mathbf{e}}_i\times\rfloor + \frac{\partial f_4}{\partial \mathbf{b}_{w_i}}\lfloor\hat{\boldsymbol{\omega}}\times\rfloor^2 - f_4(\lfloor\hat{\mathbf{e}}_i\times\rfloor\lfloor\hat{\boldsymbol{\omega}}\times + \lfloor\hat{\boldsymbol{\omega}}\times\rfloor\lfloor\hat{\mathbf{e}}_i\times\rfloor)\right)\hat{\mathbf{a}} \quad (76)$$

where

$$\frac{\partial f_3}{\partial \mathbf{b}_{w_i}} = \frac{\boldsymbol{\omega}^\top\hat{\mathbf{e}}_i(2(\text{cos}(|\hat{\boldsymbol{\omega}}|\Delta t) - 1) + (|\hat{\boldsymbol{\omega}}|\Delta t)\text{sin}(|\hat{\boldsymbol{\omega}}|\Delta t))}{|\hat{\boldsymbol{\omega}}|^4} \quad (77)$$
$$\frac{\partial f_4}{\partial \mathbf{b}_{w_i}} = \frac{\boldsymbol{\omega}^\top\hat{\mathbf{e}}_i(2(|\hat{\boldsymbol{\omega}}|\Delta t) + (|\hat{\boldsymbol{\omega}}|\Delta t)\text{cos}(|\hat{\boldsymbol{\omega}}|\Delta t) - 3\text{sin}(|\hat{\boldsymbol{\omega}}|\Delta t))}{|\hat{\boldsymbol{\omega}}|^5} \quad (78)$$

For small $\hat{\boldsymbol{\omega}}$,

$$\lim_{|w|\to 0}\frac{\partial f_3}{\partial \mathbf{b}_{w_i}} = -\hat{\boldsymbol{\omega}}_i\frac{\Delta t^4}{12} \quad (79)$$
$$\lim_{|w|\to 0}\frac{\partial f_4}{\partial \mathbf{b}_{w_i}} = \hat{\boldsymbol{\omega}}_i\frac{\Delta t^5}{60} \quad (80)$$

We now show how to derive the derivative of the rotation matrix with respect to a change in bias, i.e., $\Delta\mathbf{b}_w = \mathbf{b}_w - \bar{\mathbf{b}}_w$:

$$ {}^{\tau_1}_k\mathbf{R} = ({}^k_{\tau_1}\mathbf{R})^\top = (\exp(\lfloor\boldsymbol{\omega}_{m_1} - \bar{\mathbf{b}}_w - \Delta\mathbf{b}_w\times\rfloor\Delta t))^\top \quad (81)$$
$$\simeq (\exp(\lfloor\boldsymbol{\omega}_{m_1} - \bar{\mathbf{b}}_w\times\rfloor\Delta t)\exp(\lfloor -\mathbf{J}_{r_1}\Delta\mathbf{b}_w\times\rfloor\Delta t))^\top \quad (82)$$
$$= \exp(\lfloor\mathbf{J}_{r_1}\Delta\mathbf{b}_w\times\rfloor\Delta t)(\exp(-\lfloor\boldsymbol{\omega}_{m_1} - \bar{\mathbf{b}}_w\times\rfloor\Delta t) \quad (83)$$
$$= \exp(\lfloor\mathbf{J}_{r_1}\Delta\mathbf{b}_w\times\rfloor\Delta t){}^{\tau_1}_k\hat{\mathbf{R}} \quad (84)$$
$$\simeq (\mathbf{I}_{3\times 3} + \mathbf{J}_{r_1}\Delta\mathbf{b}_w\times\rfloor\Delta t){}^{\tau_1}_k\hat{\mathbf{R}} \quad (85)$$

where $\mathbf{J}_{r_i}$ is the right Jacobian of SO(3), see equation (2). This decomposition can be done for every measurement interval:

$$ {}^{\tau_{i+1}}_{\tau_i}\mathbf{R} \simeq (\mathbf{I}_{3\times3} + \mathbf{J}_{r_i}\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_{i+1}}_{\tau_i}\hat{\mathbf{R}} \tag{86} $$

We can now look at what happens when we compound measurements. In particular, at the second-step τ_2, we have the following:

$$ \begin{aligned} {}^{\tau_2}_{k}\mathbf{R} &= {}^{\tau_2}_{\tau_1}\mathbf{R}^{\tau_1}_{k}\mathbf{R} & (87)\\ &\simeq (\mathbf{I}_{3\times3} + \lfloor\mathbf{J}_{r_2}\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_2}_{\tau_1}\hat{\mathbf{R}}(\mathbf{I}_{3\times3} + \lfloor\mathbf{J}_{r_1}\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_1}_{k}\hat{\mathbf{R}} & (88)\\ &= (\mathbf{I}_{3\times3} + \lfloor\mathbf{J}_{r_2}\Delta\mathbf{b}_w\times\rfloor\Delta t)(\mathbf{I}_{3\times3} + \lfloor^{\tau_2}_{\tau_1}\hat{\mathbf{R}}\mathbf{J}_{r_1}\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_2}_{k}\hat{\mathbf{R}} & (89)\\ &= (\mathbf{I}_{3\times3} + \lfloor(\mathbf{J}_{r_2} + {}^{\tau_2}_{\tau_1}\hat{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_2}_{k}\hat{\mathbf{R}} & (90) \end{aligned} $$

Here we have used the property that $\mathbf{R}\lfloor\mathbf{w}\times\rfloor\mathbf{R}^\top = \lfloor\mathbf{R}\mathbf{w}\times\rfloor$ for a rotation matrix. Repeating this process for time-step τ_3 yields:

$$ \begin{aligned} {}^{\tau_3}_{k}\mathbf{R} &= {}^{\tau_3}_{\tau_2}\mathbf{R}^{\tau_2}_{k}\mathbf{R} & (91)\\ &\simeq (\mathbf{I}_{3\times3} + \lfloor\mathbf{J}_{r_3}\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_3}_{\tau_2}\hat{\mathbf{R}}(\mathbf{I}_{3\times3} + \lfloor(\mathbf{J}_{r_2} + {}^{\tau_2}_{\tau_1}\hat{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_2}_{k}\hat{\mathbf{R}} & (92)\\ &= (\mathbf{I}_{3\times3} + \lfloor\mathbf{J}_{r_3}\Delta\mathbf{b}_w\times\rfloor)(\mathbf{I}_{3\times3} + \lfloor({}^{\tau_3}_{\tau_2}\hat{\mathbf{R}}\mathbf{J}_{r_2} + {}^{\tau_3}_{\tau_1}\hat{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_3}_{k}\hat{\mathbf{R}} & (93)\\ &\simeq (\mathbf{I}_{3\times3} + \lfloor(\mathbf{J}_{r_3} + {}^{\tau_3}_{\tau_2}\hat{\mathbf{R}}\mathbf{J}_{r_2} + {}^{\tau_3}_{\tau_1}\hat{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_w\times\rfloor\Delta t)^{\tau_3}_{k}\hat{\mathbf{R}} & (94) \end{aligned} $$

We thus see the pattern developing and can write the updated rotation at any time step u as:

$$ \begin{aligned} {}^{u}_{k}\mathbf{R}_{\oplus} &= \exp(\lfloor\mathbf{J}_q(u)(\mathbf{b}_w - \bar{\mathbf{b}}_w)\times\rfloor)^{u}_{k}\mathbf{R}_{\ominus} & (95)\\ \text{with } \mathbf{J}_q(u) &= \sum_{\tau=\tau_1}^{u} {}^{u}_{\tau}\hat{\mathbf{R}}\mathbf{J}_{r_\tau}\Delta t & (96) \end{aligned} $$

Each of these values can be calculated *incrementally* by noting that:

$$ \mathbf{J}_q(u+1) = {}^{u+1}_{u}\hat{\mathbf{R}}\sum_{\tau=\tau_1}^{u} {}^{u}_{\tau}\hat{\mathbf{R}}\mathbf{J}_{r_\tau}\Delta t + \mathbf{J}_{r_{u+1}}\Delta t = {}^{u+1}_{u}\hat{\mathbf{R}}\mathbf{J}_q(u) + \mathbf{J}_{r_{u+1}}\Delta t \tag{97} $$

The derivative of every rotation with respect to the ith entry of the gyro bias, which appears in both (71) and (76) can be approximated using:

$$ \begin{aligned} {}^{u}_{k}\mathbf{R}_{\oplus} &\simeq (\mathbf{I}_{3\times3} + \lfloor\mathbf{J}_q(u)(\mathbf{b}_w - \bar{\mathbf{b}}_w)\times\rfloor)^{u}_{k}\mathbf{R}_{\ominus} & (98)\\ \frac{\partial^{u}_{k}\mathbf{R}}{\partial\mathbf{b}_{w_i}} &\approx \lfloor\mathbf{J}_q(u)\hat{\mathbf{e}}_i\times\rfloor^{u}_{k}\mathbf{R}_{\ominus} & (99)\\ \frac{\partial^{k}_{u}\mathbf{R}}{\partial\mathbf{b}_{w_i}} &\approx -{}^{k}_{u}\mathbf{R}_{\ominus}\lfloor\mathbf{J}_q(u)\hat{\mathbf{e}}_i\times\rfloor & (100) \end{aligned} $$

The total rotation after a bias update can be expressed as:

$$ \begin{aligned} {}^{k+1}_{k}\mathbf{R}_{\oplus} &= \exp(\lfloor\mathbf{J}_q(k+1)(\mathbf{b}_w - \bar{\mathbf{b}}_w)\times\rfloor)^{k+1}_{k}\mathbf{R}_{\ominus} & (101)\\ & & (102) \end{aligned} $$

Note that this is essentially the same update as seen in [2], although we parametrize opposite rotations. We use the symbol $\oplus$ to denote an estimate after update and $\ominus$ before update. The angle measurement residual can be written as:

$$ {}^{k+1}\delta\boldsymbol{\theta}_k = 2\mathbf{vec}\Big({}^{k+1}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \mathbf{quat}(\exp(-\lfloor \mathbf{J}_q(k+1)(\mathbf{b}_w - \bar{\mathbf{b}}_w)\times\rfloor)\Big) , \tag{103} $$

$$ \simeq 2\mathbf{vec}\left({}^{k+1}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \begin{bmatrix} \frac{1}{2}(\mathbf{J}_q(\mathbf{b}_w - \bar{\mathbf{b}}_w) \\ 1 \end{bmatrix}\right) , \tag{104} $$

where $\mathbf{quat}(\cdot)$ denotes the transformation of a rotation matrix to the corresponding quaternion. In the above expression, we have also used the common assumption that $(\mathbf{b}_w - \bar{\mathbf{b}}_w)$ is small. Note that we only use this approximation for the computation of Jacobians, while (105) is used for the evaluation of actual residuals.

8 Preintegration Measurement Jacobians

Our total measurement residuals can be written as:

$$ \mathbf{r} = \begin{bmatrix} {}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_k - {}^{G}\mathbf{v}_k\Delta t + \frac{1}{2}{}^{G}\mathbf{g}\Delta t^2\right) - \mathbf{J}_\alpha(\mathbf{b}_w - \bar{\mathbf{b}}_w) - \mathbf{H}_\alpha(\mathbf{b}_a - \bar{\mathbf{b}}_a) - {}^{k}\hat{\alpha}_{k+1} \\ {}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_k + {}^{G}\mathbf{g}\Delta t\right) - \mathbf{J}_\beta(\mathbf{b}_w - \bar{\mathbf{b}}_w) - \mathbf{H}_\beta(\mathbf{b}_a - \bar{\mathbf{b}}_a) - {}^{k}\hat{\beta}_{k+1} \\ 2\mathrm{vec}\Big({}^{k+1}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \mathbf{quat}(\mathrm{Exp}(-\lfloor \mathbf{J}_q(\mathbf{b}_w - \bar{\mathbf{b}}_w)\times\rfloor)\Big) \end{bmatrix} \tag{105} $$

In order to use the preintegrated measurement residuals in graph-based optimization, the corresponding Jacobians with respect to the optimization variables are necessary. To this end, we first rewrite the relative-rotation measurement residual as:

$$ {}^{k+1}\delta\theta_k = 2\mathbf{vec}\left({}^{k+1}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \bar{q}_b\right) \tag{106} $$

where $\bar{q}_b$ is the quaternion induced by a change in gyro bias and ${}^{k+1}_{k}\breve{\bar{q}}$ is the quaternion of relative rotation obtained by integrating the IMU measurements. Instead of directly computing the derivatives, the measurement Jacobian with respect to one element of the state vector can be found by perturbing the measurement function by the corresponding element. For example, the relative-rotation measurement residual is perturbed by a change in gyro bias around the current estimate (i.e., $\mathbf{b}_w - \bar{\mathbf{b}}_w = \hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w$):

$$ {}^{k+1}\delta\theta_k = 2\mathbf{vec}\left({}^{k+1}_{G}\hat{\bar{q}} \otimes {}^{k+1}_{G}\hat{\bar{q}}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w)}{2} \\ 1 \end{bmatrix}\right) \tag{107} $$

$$ =: 2\mathbf{vec}\left(\hat{\bar{q}}_r \otimes \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w)}{2} \\ 1 \end{bmatrix}\right) \tag{108} $$

$$ = 2\mathbf{vec}\left(\mathcal{L}(\hat{\bar{q}}_r) \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w)}{2} \\ 1 \end{bmatrix}\right) \tag{109} $$

$$ = 2\mathbf{vec}\left(\begin{bmatrix} \hat{q}_{r,4}\mathbf{I}_{3\times3} - \lfloor \hat{\mathbf{q}}_r \times\rfloor & \hat{\mathbf{q}}_r \\ -\hat{\mathbf{q}}_r^\top & \hat{q}_{r,4} \end{bmatrix} \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w)}{2} \\ 1 \end{bmatrix}\right) \tag{110} $$

$$ = (\hat{q}_{r,4}\mathbf{I}_{3\times3} - \lfloor \hat{\mathbf{q}}_r \times\rfloor)\mathbf{J}_q(\hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w) + \text{other terms} \tag{111} $$

So that our Jacobian with respect to a perturbance in bias is:

$$ \frac{\partial\, {}^{k+1}\delta\boldsymbol{\theta}_k}{\partial \tilde{\mathbf{b}}_w} = (\hat{q}_{r,4}\mathbf{I}_{3\times3} - \lfloor \hat{\mathbf{q}}_r \times\rfloor)\mathbf{J}_q \tag{112} $$

Similarly, the Jacobian with respect to ${}^{k+1}\delta\boldsymbol{\theta}_G$ can be found as follows:

$$
\begin{aligned}
{}^{k+1}\delta\theta_k &= 2\mathbf{vec}\left(\begin{bmatrix}\frac{1}{2}{}^{k+1}\delta\boldsymbol{\theta}_G \\ 1\end{bmatrix} \otimes {}^{k+1}_{G}\hat{\bar{q}} \otimes {}^{k}_{G}\hat{\bar{q}}^{-1} \otimes {}^{k+1}_{k}\bar{q}^{-1} \otimes \hat{\bar{q}}_b\right) && (113)\\
&= 2\mathbf{vec}\left(\begin{bmatrix}\frac{1}{2}{}^{k+1}\delta\boldsymbol{\theta}_G \\ 1\end{bmatrix} \otimes \hat{\bar{q}}_{rb}\right) && (114)\\
&= 2\mathbf{vec}\left(\mathcal{R}(\hat{q}_{rb})\begin{bmatrix}\frac{1}{2}{}^{k+1}\delta\boldsymbol{\theta}_G \\ 1\end{bmatrix}\right) && (115)\\
&= 2\mathbf{vec}\left(\begin{bmatrix}\hat{q}_{rb,4}\mathbf{I}_{3\times3} + \lfloor\hat{\mathbf{q}}_{rb}\times\rfloor & \hat{\mathbf{q}}_{rb} \\ -\hat{\mathbf{q}}_{rb}^\top & \hat{q}_{rb,4}\end{bmatrix}\begin{bmatrix}\frac{1}{2}{}^{k+1}\delta\boldsymbol{\theta}_G \\ 1\end{bmatrix}\right) && (116)\\
&= (\hat{q}_{rb,4}\mathbf{I}_{3\times3} + \lfloor\hat{\mathbf{q}}_{rb}\times\rfloor){}^{k+1}\delta\boldsymbol{\theta}_G + \text{ other terms} && (117)
\end{aligned}
$$

Yielding the Jacobian:

$$
\frac{\partial^{k+1}\delta\boldsymbol{\theta}_k}{\partial^{k+1}\delta\boldsymbol{\theta}_G} = \hat{q}_{rb,4}\mathbf{I}_{3\times3} + \lfloor\hat{\mathbf{q}}_{rb}\times\rfloor \tag{118}
$$

The Jacobian with respect to ${}^{k}\delta\boldsymbol{\theta}_G$ is given by:

$$
\begin{aligned}
{}^{k+1}\delta\theta_k &= 2\mathbf{vec}\left({}^{k+1}_{G}\hat{\bar{q}} \otimes {}^{k}_{G}\hat{\bar{q}}^{-1} \otimes \begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2} \\ 1\end{bmatrix} \otimes {}^{k+1}_{k}\bar{q}^{-1} \otimes \hat{\bar{q}}_b\right) && (119)\\
&= 2\mathbf{vec}\left(\hat{\bar{q}}_n \otimes \begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2} \\ 1\end{bmatrix} \otimes \hat{\bar{q}}_{mb}^{-1}\right) && (120)\\
&= 2\mathbf{vec}\left(\mathcal{L}(\hat{\bar{q}}_n)\mathcal{R}(\bar{q}_{mb}^{-1})\begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2} \\ 1\end{bmatrix}\right) && (121)\\
&= 2\mathbf{vec}\left(\begin{bmatrix}\hat{q}_{n,4}\mathbf{I}_{3\times3} - \lfloor\hat{\mathbf{q}}_n\times\rfloor & \hat{\mathbf{q}}_n \\ -\hat{\mathbf{q}}_n^\top & \hat{q}_{n,4}\end{bmatrix}\begin{bmatrix}\bar{q}_{mb,4}\mathbf{I}_{3\times3} - \lfloor\bar{\mathbf{q}}_{mb}\times\rfloor & -\mathbf{q}_{mb} \\ \mathbf{q}_{mb}^\top & \bar{q}_{mb,4}\end{bmatrix}\begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2} \\ 1\end{bmatrix}\right) && (122)\\
&= -((\hat{q}_{n,4}\mathbf{I}_{3\times3\times3} - \lfloor\hat{\mathbf{q}}_n\times\rfloor)(q_{mb,4}\mathbf{I}_{3\times3} - \lfloor\mathbf{q}_{mb}\times\rfloor) + \hat{\mathbf{q}}_n\mathbf{q}_{mb}^\top){}^{k}\delta\boldsymbol{\theta}_G + \text{ other terms} && (123)
\end{aligned}
$$

Which gives the Jacobian:

$$
\frac{\partial^{k+1}\delta\boldsymbol{\theta}_k}{\partial^{k}\delta\boldsymbol{\theta}_G} = -((\hat{q}_{n,4}\mathbf{I}_{3\times3} - \lfloor\hat{\mathbf{q}}_n\times\rfloor)(\bar{q}_{mb,4}\mathbf{I}_{3\times3} - \lfloor\mathbf{q}_{mb}\times\rfloor) + \hat{\mathbf{q}}_n\bar{\mathbf{q}}_{mb}^\top) \tag{124}
$$

Note than in the preceding Jacobians, we have defined several intermediate quaternions, ($\hat{\bar{q}}_r$, $\hat{\bar{q}}_{rb}$, $\hat{\bar{q}}_n$, and $\hat{\bar{q}}_{mb}$) for ease of notation. Following the same methodology, we can find the Jacobians of the $\boldsymbol{\alpha}$ measurement with respect to the position, velocity and bias.

$$
\begin{aligned}
{}^{k}\boldsymbol{\alpha}_{k+1} &= {}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_k - {}^{G}\mathbf{v}_k\Delta t + \frac{1}{2}{}^{G}\mathbf{g}\Delta t^2\right) - \mathbf{J}_\alpha(\mathbf{b}_w - \bar{\mathbf{b}}_w) - \mathbf{H}_\alpha(\mathbf{b}_a - \bar{\mathbf{b}}_a)\\
&\simeq \left(\mathbf{I}_{3\times3} - \lfloor{}^{k}\delta\boldsymbol{\theta}_G\times\rfloor\right){}^{k}_{G}\hat{\mathbf{R}}\Big({}^{G}\hat{\mathbf{p}}_{k+1} + {}^{G}\tilde{\mathbf{p}}_{k+1} - {}^{G}\hat{\mathbf{p}}_k - {}^{G}\tilde{\mathbf{p}}_k - {}^{G}\hat{\mathbf{v}}_k\Delta t - {}^{G}\tilde{\mathbf{v}}_k\Delta t\\
&\qquad + \frac{1}{2}{}^{G}\mathbf{g}\Delta t^2\Big) - \mathbf{J}_\alpha\left(\hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w\right) - \mathbf{H}_\alpha\left(\hat{\mathbf{b}}_a + \tilde{\mathbf{b}}_a - \bar{\mathbf{b}}_a\right) && (125)
\end{aligned}
$$

Then the following Jacobians immediately becomes available:

$$\frac{\partial^k \boldsymbol{\alpha}_{k+1}}{\partial^k \delta\boldsymbol{\theta}_G} = \left\lfloor {}^k_G\hat{\mathbf{R}} \left({}^G\hat{\mathbf{p}}_{k+1} - {}^G\hat{\mathbf{p}}_k - {}^G\hat{\mathbf{v}}_k \Delta t + \frac{1}{2} {}^G\mathbf{g}\Delta t^2 \right) \times \right\rfloor \tag{126}$$

$$\frac{\partial^k \boldsymbol{\alpha}_{k+1}}{\partial^G \mathbf{p}_k} = -{}^k_G\hat{\mathbf{R}} \tag{127}$$

$$\frac{\partial^k \boldsymbol{\alpha}_{k+1}}{\partial^G \mathbf{p}_{k+1}} = {}^k_G\hat{\mathbf{R}} \tag{128}$$

$$\frac{\partial^k \boldsymbol{\alpha}_{k+1}}{\partial^G \mathbf{v}_k} = -{}^k_G\hat{\mathbf{R}}\Delta t \tag{129}$$

$$\frac{\partial^k \boldsymbol{\alpha}_{k+1}}{\partial \tilde{\mathbf{b}}_w} = -\mathbf{J}_\alpha \tag{130}$$

$$\frac{\partial^k \boldsymbol{\alpha}_{k+1}}{\partial \tilde{\mathbf{b}}_a} = -\mathbf{H}_\alpha \tag{131}$$

Similarly, we can write our $\boldsymbol{\beta}$ measurement as:

$$\begin{aligned} {}^k\boldsymbol{\beta}_{k+1} &= {}^k_G\mathbf{R} \left({}^G\mathbf{v}_{k+1} - {}^G\mathbf{v}_k + {}^G\mathbf{g}\Delta t \right) - \mathbf{J}_\beta(\mathbf{b}_w - \bar{\mathbf{b}}_w) \\ &\simeq (\mathbf{I}_{3\times 3} - \lfloor {}^k\delta\boldsymbol{\theta}_G \times \rfloor) {}^k_G\hat{\mathbf{R}} \left({}^G\hat{\mathbf{v}}_{k+1} + {}^G\tilde{\mathbf{v}}_{k+1} - {}^G\hat{\mathbf{v}}_k - {}^G\tilde{\mathbf{v}}_k + {}^G\mathbf{g}\Delta t \right) \\ &\qquad - \mathbf{J}_\beta(\hat{\mathbf{b}}_w + \tilde{\mathbf{b}}_w - \bar{\mathbf{b}}_w) - \mathbf{H}_\beta(\hat{\mathbf{b}}_a + \tilde{\mathbf{b}}_a - \bar{\mathbf{b}}_a) \end{aligned} \tag{132}$$

which leads to the following Jacobians:

$$\frac{\partial^k \boldsymbol{\beta}_{k+1}}{\partial^k \delta\boldsymbol{\theta}_G} = \left\lfloor {}^k_G\hat{\mathbf{R}}({}^G\hat{\mathbf{v}}_{k+1} - {}^G\hat{\mathbf{v}}_k + {}^G\mathbf{g}\Delta t) \times \right\rfloor \tag{133}$$

$$\frac{\partial^k \boldsymbol{\beta}_{k+1}}{\partial^G \mathbf{v}_k} = -{}^k_G\hat{\mathbf{R}} \tag{134}$$

$$\frac{\partial^k \boldsymbol{\beta}_{k+1}}{\partial^G \mathbf{v}_{k+1}} = {}^k_G\hat{\mathbf{R}} \tag{135}$$

$$\frac{\partial^k \boldsymbol{\beta}_{k+1}}{\partial \tilde{\mathbf{b}}_w} = -\mathbf{J}_\beta \tag{136}$$

$$\frac{\partial^k \boldsymbol{\beta}_{k+1}}{\partial \tilde{\mathbf{b}}_w} = -\mathbf{H}_\beta \tag{137}$$

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