Continuous Preintegration Theory for Graph-based Visual-Inertial Navigation

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1 Preliminaries

1.1 System State

The IMU state of an aided inertial navigation system at time step k is given by [1]:

$$\mathbf{x}_k = \begin{bmatrix} {}^k_G\bar{q}^\top & \mathbf{b}_{\omega_k}^\top & {}^G\mathbf{v}_k^\top & \mathbf{b}_{a_k}^\top & {}^G\mathbf{p}_k^\top \end{bmatrix}^\top \tag{1}$$

where ${}^k_G\bar{q}$ is the unit quaternion of JPL form parameterizing the rotation ${}^k_G\mathbf{R}$ from the global frame $\{G\}$ to the current local frame $\{k\}$ [2], $\mathbf{b}_{\omega_k}$ and $\mathbf{b}_{a_k}$ are the gyroscope and accelerometer biases, and ${}^G\mathbf{v}_k$ and ${}^G\mathbf{p}_k$ are the velocity and position of the IMU expressed in the global frame, respectively. The error state corresponding to Equation (1) is given by:

$$\delta\mathbf{x}_k = \begin{bmatrix} {}^k\delta\boldsymbol{\theta}_G^\top & \delta\mathbf{b}_{\omega_k}^\top & {}^G\delta\mathbf{v}_k^\top & \delta\mathbf{b}_{a_k}^\top & {}^G\delta\mathbf{p}_k^\top \end{bmatrix}^\top \tag{2}$$

The relationship between the vector quantities with true value $\mathbf{v}$, mean value $\hat{\mathbf{v}}$, and error state $\delta\mathbf{v}$ takes the form $\mathbf{v} = \hat{\mathbf{v}} + \delta\mathbf{v}$. For quaternions in JPL convention, with true value $\bar{q}$, mean value $\hat{\bar{q}}$, and error state $\delta\boldsymbol{\theta}$, we have:

$$\bar{q} = \begin{bmatrix} \frac{\delta\boldsymbol{\theta}}{2} \\ 1 \end{bmatrix} \otimes \hat{\bar{q}} \tag{3}$$

where $\otimes$ is the quaternion multiplication. We write these relationships compactly as $\mathbf{x} = \hat{\mathbf{x}} \boxplus \delta\mathbf{x}$ and $\delta\mathbf{x} = \mathbf{x} \boxminus \hat{\mathbf{x}}$.

1.2 Graph Optimization

Given a set of measurements with residuals $\mathbf{e}_i$ and information matrices $\boldsymbol{\Lambda}_i$, we solve the following maximum a posteriori (MAP) problem to estimate our state [3]:

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \sum_i \frac{1}{2} ||\mathbf{e}_i(\mathbf{x})||^2_{\boldsymbol{\Lambda}_i} \tag{4}$$

We solve this MAP problem through iterative linearization, where for each iteration we solve the following problem:

$$\delta\hat{\mathbf{x}} = \underset{\delta\mathbf{x}}{\operatorname{argmin}} \sum_i \frac{1}{2} ||\mathbf{e}_i(\hat{\mathbf{x}}) + \mathbf{J}_i\delta\mathbf{x}||^2_{\boldsymbol{\Lambda}_i} \tag{5}$$

$$\mathbf{J}_i = \frac{\partial \mathbf{e}_i(\hat{\mathbf{x}} \boxplus \delta\mathbf{x})}{\partial\delta\mathbf{x}}\Big|_{\delta\mathbf{x}=\mathbf{0}} \tag{6}$$

It is clear that in order to use this formulation in estimation, we must define the appropriate measurement residuals $\mathbf{e}_i$, Jacobians $\mathbf{J}_i$, and information matrices $\boldsymbol{\Lambda}_i$. In this technical report we show how to optimally utilize IMU measurements in this formulation through the use of continuous preintegration.

2 Continuous Preintegration

An IMU attached to the robot collects inertial readings of the underlying state dynamics. In particular, the sensor receives angular velocity $\boldsymbol{\omega}_m$ and local linear acceleration $\mathbf{a}_m$ measurements which relate to the corresponding true values $\boldsymbol{\omega}$ and $\mathbf{a}$ as follows:

$$\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_\omega + \mathbf{n}_\omega \tag{7}$$

$$\mathbf{a}_m = \mathbf{a} + {}^I_G\mathbf{R}\,{}^G\mathbf{g} + \mathbf{b}_a + \mathbf{n}_a \tag{8}$$

where ${}^G\mathbf{g} = [0\ 0\ 9.81]^\top$ is the global gravity, noting that the gravity is slightly different in different parts of the globe, and ${}^I_G\mathbf{R}$ is the rotation from the global frame to the instantaneous local inertial frame. The measurements are corrupted both by the time-varying biases $\mathbf{b}_\omega$ and $\mathbf{b}_a$ (which must be co-estimated with the state), and the zero-mean white Gaussian noises $\mathbf{n}_\omega$ and $\mathbf{n}_a$. The standard dynamics of the IMU state is given by [4]:

$${}^I_G\dot{\bar{q}} = \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\omega}_m - \mathbf{b}_\omega - \mathbf{n}_\omega)\,{}^I_G\bar{q} \tag{9}$$

$$\dot{\mathbf{b}}_\omega = \mathbf{n}_{\omega b} \tag{10}$$

$${}^G\dot{\mathbf{v}}_I = {}^G_I\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) - {}^G\mathbf{g} \tag{11}$$

$$\dot{\mathbf{b}}_a = \mathbf{n}_{ab} \tag{12}$$

$${}^G\dot{\mathbf{p}}_I = {}^G\mathbf{v}_I \tag{13}$$

where

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) = \begin{bmatrix} -\lfloor\boldsymbol{\omega}\rfloor & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^\top & 0 \end{bmatrix} \tag{14}$$

3 Standard IMU Processing

Given a series of IMU measurements, $\mathcal{I}$, collected over a time interval $[t_k, t_{k+1}]$, the standard (graph-based) IMU processing considers the following propagation function:

$$\mathbf{x}_{k+1} = \mathbf{g}\left(\mathbf{x}_k, \mathcal{I}, \mathbf{n}\right) \tag{15}$$

That is, the future state at time step $k+1$ is a function of the current state at step k, the IMU measurements $\mathcal{I}$, and the corresponding measurement noise $\mathbf{n}$. Conditioning on the current state, the expected value of the next state is found by evaluating the propagation function with zero noise:

$$\breve{\mathbf{x}}_{k+1} = \mathbf{g}\left(\mathbf{x}_k, \mathcal{I}, \mathbf{0}\right) \tag{16}$$

which implies that we perform integration of the state dynamics in the absence of noise. The residual for use in batch optimization of this propagation now constrains the start and end states of the interval and is given by:

$$c_{IMU}(\mathbf{x}) = \frac{1}{2}\left|\left|\mathbf{x}_{k+1} \boxminus \breve{\mathbf{x}}_{k+1}\right|\right|^2_{\mathbf{Q}_k^{-1}} \tag{17}$$

$$= \frac{1}{2}\left|\left|\mathbf{x}_{k+1} \boxminus \mathbf{g}\left(\mathbf{x}_k, \mathcal{I}, \mathbf{0}\right)\right|\right|^2_{\mathbf{Q}_k^{-1}} \tag{18}$$

where $\mathbf{Q}_k$ is the linearized, discrete-time noise covariance computed from the IMU noise characterization and is a *function of the state*. This noise covariance matrix and the propagation function can be found by the integration of Equations (9)-(13) and their associated error state dynamics, to which we refer the reader to [2, 1].

It is clear from Equation (16) that we need constantly re-evaluate the propagation function $\mathbf{g}(\cdot)$ and the residual covariance $\mathbf{Q}_k$ whenever the linearization point (state estimate) changes. The high frequency nature of the IMU sensors and the complexity of the propagation function and the noise covariance can make direct incorporation of IMU data in real-time graph-based SLAM prohibitively expensive.

4 Model 1: Piecewise Constant Measurements

IMU preintegration seeks to directly reduce the computational complexity of incorporating inertial measurements by removing the need to re-integrate the propagation function and noise covariance. This is achieved by processing IMU measurements in a *local* frame of reference, yielding measurements that are, in contrast to Equation (16), independent of the state [5].

Specifically, by denoting $\Delta T = t_{k+1} - t_k$, we have the following relationship between a series of IMU measurements, the start state, and the resulting end state [6]:

$$^G\mathbf{p}_{k+1} = {}^G\mathbf{p}_k + {}^G\mathbf{v}_k\Delta T - \frac{1}{2}{}^G\mathbf{g}\Delta T^2 + {}^G_k\mathbf{R}\int_{t_k}^{t_{k+1}}\int_{t_k}^{s} {}^k_u\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) du ds \tag{19}$$

$$^G\mathbf{v}_{k+1} = {}^G\mathbf{v}_k - {}^G\mathbf{g}\Delta T + {}^G_k\mathbf{R}\int_{t_k}^{t_{k+1}} {}^k_u\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) du \tag{20}$$

$$^{k+1}_G\mathbf{R} = {}^{k+1}_k\mathbf{R}\,{}^k_G\mathbf{R} \tag{21}$$

$$\mathbf{b}_{\omega_{k+1}} = \mathbf{b}_{\omega_k} + \int_{t_k}^{t_{k+1}} \mathbf{n}_{\omega b}\; du \tag{22}$$

$$\mathbf{b}_{a_{k+1}} = \mathbf{b}_{a_k} + \int_{t_k}^{t_{k+1}} \mathbf{n}_{ab}\; du \tag{23}$$

From the above, we define the following preintegrated IMU measurements:

$$^k\boldsymbol{\alpha}_{k+1} = \int_{t_k}^{t_{k+1}}\int_{t_k}^{s} {}^k_u\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) du ds \tag{24}$$

$$^k\boldsymbol{\beta}_{k+1} = \int_{t_k}^{t_{k+1}} {}^k_u\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) du \tag{25}$$

where along with these preintegrated inertial measurements the preintegrated relative-orientation measurement ${}^{k+1}_k\bar{q}$ (or ${}^{k+1}_k\mathbf{R}$) can be obtained from the integration of the gyro measurements.

To remove the dependencies of the above preintegrated measurements on the true biases, we linearize about the current bias estimates at time step t_k, $\mathbf{b}^\star_{a_k}$ and $\mathbf{b}^\star_{\omega_k}$. Defining $\Delta\mathbf{b} = \mathbf{b} - \mathbf{b}^\star$, we have:

$$\begin{aligned}{}^k_G\mathbf{R}\left({}^G\mathbf{p}_{k+1} - {}^G\mathbf{p}_k - {}^G\mathbf{v}_k\Delta T + \frac{1}{2}{}^G\mathbf{g}\Delta T^2\right) \simeq \\ {}^k\boldsymbol{\alpha}_{k+1}\left(\mathbf{b}^\star_{\omega_k}, \mathbf{b}^\star_{a_k}\right) + \left.\frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{b}_\omega}\right|_{\mathbf{b}^\star_{\omega_k}}\Delta\mathbf{b}_\omega + \left.\frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{b}_a}\right|_{\mathbf{b}^\star_{a_k}}\Delta\mathbf{b}_a\end{aligned} \tag{26}$$

$${}^k_G\mathbf{R}\left({}^G\mathbf{v}_{k+1} - {}^G\mathbf{v}_k + {}^G\mathbf{g}\Delta T\right) \simeq {}^k\boldsymbol{\beta}_{k+1}\left(\mathbf{b}^\star_{\omega_k}, \mathbf{b}^\star_{a_k}\right) + \left.\frac{\partial\boldsymbol{\beta}}{\partial\mathbf{b}_\omega}\right|_{\mathbf{b}^\star_{\omega_k}}\Delta\mathbf{b}_\omega + \left.\frac{\partial\boldsymbol{\beta}}{\partial\mathbf{b}_a}\right|_{\mathbf{b}^\star_{a_k}}\Delta\mathbf{b}_a \tag{27}$$

$${}^{k+1}_G\mathbf{R}\,{}^k_G\mathbf{R}^\top \simeq \mathbf{R}\left(\left.\frac{\partial\mathbf{R}}{\partial\mathbf{b}_\omega}\right|_{\mathbf{b}^\star_{\omega_k}}\Delta\mathbf{b}_\omega\right){}^{k+1}_k\mathbf{R}\left(\mathbf{b}^\star_{\omega_k}\right) \tag{28}$$

Note that Equations (26) and (27) are simple Taylor series expansions for our ${}^k\boldsymbol{\alpha}_{k+1}$ and ${}^k\boldsymbol{\beta}_{k+1}$ measurements, while Equation (28) models an additional rotation induced due to a change of the linearization point (estimate) of the gyro bias [6, 7].

The preintegrated measurement's mean values, ${}^k\breve{\boldsymbol{\alpha}}_{k+1}$, ${}^k\breve{\boldsymbol{\beta}}_{k+1}$, and ${}^{k+1}_{k}\breve{\bar{q}}$, must be computed for use in graph optimization. It is important to note that current preintegration methods [5, 7, 8] are all based on discrete integration of the measurement dynamics through Euler or midpoint integration. In particular, the discrete approximation used by Forster et al. [7] in fact corresponds to a piecewise constant *global acceleration* model. By contrast, we here offer *closed-form* solutions for the measurement means under the assumptions of piecewise constant *measurements* and of piecewise constant *local acceleration* (in Section 5).

4.1 Measurement Mean

We now derive the closed-from solutions for ${}^{k+1}_{k}\breve{\bar{q}}$, ${}^k\breve{\boldsymbol{\alpha}}_{k+1}$, ${}^k\breve{\boldsymbol{\beta}}_{k+1}$. In particular, the quaternion ${}^{\tau+1}_{\tau}\breve{\bar{q}}$ can be found using the zeroth order quaternion integrator [2]. This can be compounded successively to get the final measurement mean ${}^{k+1}_{k}\breve{\bar{q}}$. In a similar fashion, we can find, in closed form, ${}^k\breve{\boldsymbol{\alpha}}_{\tau+1}$ and ${}^k\breve{\boldsymbol{\beta}}_{\tau+1}$. We have derived the following continuous time linear system:

$$\begin{bmatrix} {}^k\dot{\breve{\boldsymbol{\alpha}}}_u \\ {}^k\dot{\breve{\boldsymbol{\beta}}}_u \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_u \\ {}^k\breve{\boldsymbol{\beta}}_u \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ {}^k_u\breve{\mathbf{R}} \end{bmatrix} (\mathbf{a}_m - \mathbf{b}^{\star}_{a_k}) \tag{29}$$

The solution of this linear dynamical system is given by:

$$\begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_{\tau+1} \\ {}^k\breve{\boldsymbol{\beta}}_{\tau+1} \end{bmatrix} = \boldsymbol{\Phi}(t_{\tau+1}, t_\tau) \begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_\tau \\ {}^k\breve{\boldsymbol{\beta}}_\tau \end{bmatrix} + \int_{t_\tau}^{t_{\tau+1}} \boldsymbol{\Phi}(t_{\tau+1}, u) \begin{bmatrix} \mathbf{0} \\ {}^k_u\breve{\mathbf{R}} \end{bmatrix} (\mathbf{a}_m - \mathbf{b}^{\star}_{a_k})\, du \tag{30}$$

where the state-transition matrix $\boldsymbol{\Phi}(t_{\tau+1}, t_\tau)$ is given by the matrix exponential:

$$\boldsymbol{\Phi}(t_{\tau+1}, t_\tau) = \exp\left(\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Delta t \right) = \mathbf{I} + \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Delta t + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \Delta t^2 = \begin{bmatrix} \mathbf{I} & \mathbf{I}\Delta t \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{31}$$

where $\Delta t = t_{\tau+1} - t_\tau$. Substituting the state-transition into Equation (30) yields:

$$\begin{aligned}
\begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_{\tau+1} \\ {}^k\breve{\boldsymbol{\beta}}_{\tau+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{I}\Delta t \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_\tau \\ {}^k\breve{\boldsymbol{\beta}}_\tau \end{bmatrix} + \int_{t_\tau}^{t_{\tau+1}} \begin{bmatrix} \mathbf{I} & \mathbf{I}(t_{\tau+1} - u) \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ {}^k_u\breve{\mathbf{R}} \end{bmatrix} \hat{\mathbf{a}}\, du && (32) \\
&= \begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_\tau + {}^k\breve{\boldsymbol{\beta}}_\tau \Delta t \\ {}^k\breve{\boldsymbol{\beta}}_\tau \end{bmatrix} + \int_{t_\tau}^{t_{\tau+1}} \begin{bmatrix} (t_{\tau+1} - u)\, {}^k_u\breve{\mathbf{R}}\hat{\mathbf{a}} \\ {}^k_u\breve{\mathbf{R}}\hat{\mathbf{a}} \end{bmatrix} du && (33) \\
&= \begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_\tau + {}^k\breve{\boldsymbol{\beta}}_\tau \Delta t \\ {}^k\breve{\boldsymbol{\beta}}_\tau \end{bmatrix} + \begin{bmatrix} {}^{k}_{\tau+1}\breve{\mathbf{R}} \int_{t_\tau}^{t_{\tau+1}} (t_{\tau+1} - u)\, {}^{\tau+1}_{u}\breve{\mathbf{R}}\hat{\mathbf{a}}\, du \\ {}^{k}_{\tau+1}\breve{\mathbf{R}} \int_{t_\tau}^{t_{\tau+1}} {}^{\tau+1}_{u}\breve{\mathbf{R}}\hat{\mathbf{a}}\, du \end{bmatrix} && (34)
\end{aligned}$$

where $\hat{\mathbf{a}} = \mathbf{a}_m - \mathbf{b}^{\star}_{a_k}$. Using $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}_m - \mathbf{b}^{\star}_{\omega_k}$ and $\delta t = (t_{\tau+1} - u)$, and the Rodrigues' formula (35) we have:

$${}^{\tau+1}_{u}\mathbf{R} = \exp(-\lfloor \hat{\boldsymbol{\omega}} \rfloor \delta t) = \mathbf{I} - \frac{\sin(|\hat{\boldsymbol{\omega}}|\delta t)}{|\hat{\boldsymbol{\omega}}|} \lfloor \hat{\boldsymbol{\omega}} \rfloor + \frac{1 - \cos(|\hat{\boldsymbol{\omega}}|\delta t)}{|\hat{\boldsymbol{\omega}}|^2} \lfloor \hat{\boldsymbol{\omega}} \rfloor^2 \tag{35}$$

$$
\begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_{\tau+1} \\ {}^k\breve{\boldsymbol{\beta}}_{\tau+1} \end{bmatrix} = \begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_{\tau} + {}^k\breve{\boldsymbol{\beta}}_{\tau}\Delta t \\ {}^k\breve{\boldsymbol{\beta}}_{\tau} \end{bmatrix} + \begin{bmatrix} {}^{k}_{\tau+1}\breve{\mathbf{R}} \int_0^{\Delta t} (\delta t)(\mathbf{I} - \frac{\sin(|\hat{\boldsymbol{\omega}}|(\delta t))}{|\hat{\boldsymbol{\omega}}|}\lfloor \hat{\boldsymbol{\omega}} \rfloor + \frac{1-\cos(|\hat{\boldsymbol{\omega}}|\delta t)}{|\hat{\boldsymbol{\omega}}|^2}\lfloor \hat{\boldsymbol{\omega}} \rfloor^2)(\hat{\mathbf{a}}) d\delta t \\ {}^{k}_{\tau+1}\breve{\mathbf{R}} \int_0^{\Delta t} (\mathbf{I} - \frac{\sin(|\hat{\boldsymbol{\omega}}|(\delta t))}{|\hat{\boldsymbol{\omega}}|}\lfloor \hat{\boldsymbol{\omega}} \rfloor + \frac{1-\cos(|\hat{\boldsymbol{\omega}}|\delta t)}{|\hat{\boldsymbol{\omega}}|^2}\lfloor \hat{\boldsymbol{\omega}} \rfloor^2)(\hat{\mathbf{a}}) d\delta t \end{bmatrix} \tag{36}
$$

$$
\begin{aligned}
&= \begin{bmatrix} {}^k\breve{\boldsymbol{\alpha}}_{\tau} + {}^k\breve{\boldsymbol{\beta}}_{\tau}\Delta t \\ {}^k\breve{\boldsymbol{\beta}}_{\tau} \end{bmatrix} \\
&+ \begin{bmatrix} {}^{k}_{\tau+1}\breve{\mathbf{R}}(\frac{(\Delta t^2)}{2}\mathbf{I} + \frac{|\hat{\boldsymbol{\omega}}|\Delta t\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|w|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor \hat{\boldsymbol{\omega}} \rfloor + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t)^2 - 2\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - 2(|\hat{\boldsymbol{\omega}}|\Delta t)\sin(|\hat{\boldsymbol{\omega}}|\Delta t) + 2}{2|\hat{\boldsymbol{\omega}}|^4}\lfloor \hat{\boldsymbol{\omega}} \rfloor^2)(\hat{\mathbf{a}}) \\ {}^{k}_{\tau+1}\breve{\mathbf{R}}(\Delta t\mathbf{I} - \frac{1-\cos(|\hat{\boldsymbol{\omega}}|(\Delta t))}{|\hat{\boldsymbol{\omega}}|^2}\lfloor \hat{\boldsymbol{\omega}} \rfloor + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|\hat{\boldsymbol{\omega}}|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor \hat{\boldsymbol{\omega}} \rfloor^2)(\hat{\mathbf{a}}) \end{bmatrix}
\end{aligned} \tag{37}
$$

4.2 State-Transition Matrix

In order to use the derived continuous preintegration measurement means, we must have the covariances associated with their error quantities. To do this, we examine the time evolution of the corresponding error states:

$$
{}^k\delta\dot{\boldsymbol{\alpha}}_u = {}^k\delta\boldsymbol{\beta}_u \tag{38}
$$

$$
{}^k\delta\dot{\boldsymbol{\beta}}_u = {}^k_u\breve{\mathbf{R}}\left(\mathbf{I} + \lfloor {}^u\delta\boldsymbol{\theta}_k \rfloor\right)\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a\right) - {}^k_u\breve{\mathbf{R}}\left(\mathbf{a}_m - \mathbf{b}^{\star}_{a_k}\right) \tag{39}
$$

$$
= {}^k_u\breve{\mathbf{R}}\left(-\widetilde{\mathbf{b}}_a - \mathbf{n}_a\right) + {}^k_u\breve{\mathbf{R}}\lfloor {}^u\delta\boldsymbol{\theta}_k \rfloor\left(\mathbf{a}_m - \mathbf{b}^{\star}_{a_k}\right) \tag{40}
$$

$$
{}^u\delta\dot{\boldsymbol{\theta}}_k = -\left\lfloor\left(\hat{\boldsymbol{\omega}} - \mathbf{b}^{\star}_{\omega_k}\right)\right\rfloor {}^u\delta\boldsymbol{\theta}_k - \widetilde{\mathbf{b}}_w - \mathbf{n}_w \tag{41}
$$

where we have used the standard error associated with JPL-convention quaternions, ${}^u_k\bar{q} = \delta\bar{q} \otimes {}^u_k\breve{\bar{q}}$, and $\delta\bar{q} \simeq [(\delta\boldsymbol{\theta}/2)^\top \ 1]^\top$. This yields the following linearized system describing our error states:

$$
\begin{bmatrix} {}^u\delta\dot{\boldsymbol{\theta}}_k \\ \dot{\widetilde{\mathbf{b}}}_\omega \\ {}^k\delta\dot{\boldsymbol{\beta}}_u \\ \dot{\widetilde{\mathbf{b}}}_a \\ {}^k\delta\dot{\boldsymbol{\alpha}}_u \end{bmatrix} = \begin{bmatrix} -\lfloor\hat{\boldsymbol{\omega}}\rfloor & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -{}^k_u\breve{\mathbf{R}}\lfloor\hat{\mathbf{a}}\rfloor & \mathbf{0} & \mathbf{0} & -{}^k_u\breve{\mathbf{R}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} {}^u\delta\boldsymbol{\theta}_k \\ \widetilde{\mathbf{b}}_\omega \\ {}^k\delta\boldsymbol{\beta}_u \\ \widetilde{\mathbf{b}}_a \\ {}^k\delta\boldsymbol{\alpha}_u \end{bmatrix} + \begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -{}^k_u\breve{\mathbf{R}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{n}_\omega \\ \mathbf{n}_{\omega b} \\ \mathbf{n}_a \\ \mathbf{n}_{ab} \end{bmatrix} \tag{42}
$$

$$
\Rightarrow \quad \dot{\mathbf{r}} = \mathbf{F}\mathbf{r} + \mathbf{G}\mathbf{n} \tag{43}
$$

As compared to the original preintegration paper [6], the above system incorporates bias errors that captures the drift in a given bias over an interval. Note that these bias error terms $\widetilde{\mathbf{b}}_\omega$ and $\widetilde{\mathbf{b}}_a$, describe the deviation of the bias over the interval due to the random-walk drift, rather than the error of the current bias estimate. The discrete state transition matrix can be found by solving the following differential equation:

$$
\dot{\boldsymbol{\Phi}}(t_u, t_\tau) = \mathbf{F}(u) \ \boldsymbol{\Phi}(t_u, t_\tau) \tag{44}
$$

$$
\boldsymbol{\Phi}(t_\tau, t_\tau) = \mathbf{I}_{15\times15} \tag{45}
$$

4.3 Discrete Covariance Propagation

Using the expressions for the state-transition matrix, the covariance propagation for our preintegrated measurements takes the form:

$$
\mathbf{P}_k = \mathbf{0}_{15\times15} \tag{46}
$$

$$
\mathbf{P}_{\tau+1} = \boldsymbol{\Phi}(t_{\tau+1}, t_\tau) \ \mathbf{P}_\tau \ \boldsymbol{\Phi}(t_{\tau+1}, t_\tau)^\top + \mathbf{Q}_\tau \tag{47}
$$

$$
\mathbf{Q}_\tau = \int_{t_\tau}^{t_{\tau+1}} \boldsymbol{\Phi}(t_{\tau+1}, u)\mathbf{G}(u)\mathbf{Q}_c\mathbf{G}(u)^\top\boldsymbol{\Phi}(t_{\tau+1}, u)^\top du \tag{48}
$$

In practical applications, the state transition matrix $\mathbf{\Phi}(t_{\tau+1}, t_\tau)$ and discrete measurement covariance $\mathbf{Q}_\tau$ can be numerically integrated.

4.4 Acceleration Bias Jacobians

We have derived the closed-form measurement update for ${}^k\breve{\boldsymbol{\alpha}}_\tau$ and ${}^k\breve{\boldsymbol{\beta}}_\tau$ (see Equation (37)). We need only compute the gradients of our closed-form update expressions with respect to changes in bias. In particular, since each update term is linear in the estimated acceleration, $\hat{\mathbf{a}} = \mathbf{a}_m - \mathbf{b}^\star_{a_k}$, we can find the bias Jacobians of ${}^k\boldsymbol{\alpha}_{k+1}$ and ${}^k\boldsymbol{\beta}_{k+1}$ with respect to $\mathbf{b}_a$ as follows:

$$\begin{bmatrix} \frac{\partial \boldsymbol{\alpha}}{\partial \mathbf{b}_a} \\ \frac{\partial \boldsymbol{\beta}}{\partial \mathbf{b}_a} \end{bmatrix} =: \begin{bmatrix} \mathbf{H}_\alpha(\tau+1) \\ \mathbf{H}_\beta(\tau+1) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_\alpha(\tau) + \mathbf{H}_\beta(\tau)\Delta t \\ \mathbf{H}_\beta(\tau) \end{bmatrix} \tag{49}$$
$$- \begin{bmatrix} {}^k_{\tau+1}\breve{\mathbf{R}}\left(\frac{\Delta t^2}{2}\mathbf{I}_{3\times3} + \frac{|\hat{\boldsymbol{\omega}}|\Delta t\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|\hat{\boldsymbol{\omega}}|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor\hat{\boldsymbol{\omega}}\rfloor + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t)^2 - 2\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - 2(|\hat{\boldsymbol{\omega}}|\Delta t)\sin(|\hat{\boldsymbol{\omega}}|\Delta t) + 2}{2|\hat{\boldsymbol{\omega}}|^4}\lfloor\hat{\boldsymbol{\omega}}\rfloor^2\right) \\ {}^k_{\tau+1}\breve{\mathbf{R}}\left(\Delta t\mathbf{I}_{3\times3} - \frac{1-\cos(|\hat{\boldsymbol{\omega}}|(\Delta t))}{|\hat{\boldsymbol{\omega}}|^2}\lfloor\hat{\boldsymbol{\omega}}\rfloor + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|\boldsymbol{\omega}|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor\hat{\boldsymbol{\omega}}\rfloor^2\right) \end{bmatrix}$$

and for small values of $\hat{\boldsymbol{\omega}}$ we have:

$$\lim_{|\hat{\boldsymbol{\omega}}|\to 0} \begin{bmatrix} \mathbf{H}_\alpha(\tau+1) \\ \mathbf{H}_\beta(\tau+1) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_\alpha(\tau) + \mathbf{H}_\beta(\tau)\Delta t \\ \mathbf{H}_\beta(\tau) \end{bmatrix} - \begin{bmatrix} {}^k_{\tau+1}\breve{\mathbf{R}}(\frac{\Delta t^2}{2}\mathbf{I} - \frac{\Delta t^3}{3}\lfloor\hat{\boldsymbol{\omega}}\rfloor + \frac{\Delta t^4}{8}\lfloor\hat{\boldsymbol{\omega}}\rfloor^2) \\ {}^k_{\tau+1}\breve{\mathbf{R}}(\Delta t\mathbf{I} - \frac{\Delta t^2}{2}\lfloor\hat{\boldsymbol{\omega}}\rfloor + \frac{\Delta t^3}{6}\lfloor\hat{\boldsymbol{\omega}}\rfloor^2) \end{bmatrix} \tag{50}$$

4.5 Gyro Bias Jacobians

We find the derivatives of ${}^k\boldsymbol{\alpha}_{\tau+1}$ and ${}^k\boldsymbol{\beta}_{\tau+1}$ with respect to each entry of the gyro bias by taking the derivative with respect to each gyro bias entry. We seek to find the following entries:

$$\mathbf{J}_\alpha = \begin{bmatrix} \frac{\partial^k\boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{\omega_1}} & \frac{\partial^k\boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{\omega_2}} & \frac{\partial^k\boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{\omega_3}} \end{bmatrix} \tag{51}$$
$$\mathbf{J}_\beta = \begin{bmatrix} \frac{\partial^k\boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{\omega_1}} & \frac{\partial^k\boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{\omega_2}} & \frac{\partial^k\boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{\omega_3}} \end{bmatrix} \tag{52}$$

For each of the above entries, from Equation (37), we have the following:

$$\begin{aligned} \frac{\partial^k\boldsymbol{\alpha}_{\tau+1}}{\partial \mathbf{b}_{\omega_i}} = & \frac{\partial^k\boldsymbol{\alpha}_\tau}{\partial \mathbf{b}_{\omega_i}} + \frac{\partial^k\boldsymbol{\beta}_\tau\Delta t}{\partial \mathbf{b}_{\omega_i}} \\ & + \frac{\partial}{\partial \mathbf{b}_{w_i}}\Bigg({}^k_{\tau+1}\mathbf{R}(\frac{(\Delta t^2)}{2}\mathbf{I}_{3\times3} + \frac{|\hat{\boldsymbol{\omega}}|\Delta t\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - \sin(|\hat{\boldsymbol{\omega}}|\Delta t)}{|\hat{\boldsymbol{\omega}}|^3}\lfloor\hat{\boldsymbol{\omega}}\rfloor \\ & + \frac{(|\hat{\boldsymbol{\omega}}|\Delta t)^2 - 2\cos(|\hat{\boldsymbol{\omega}}|\Delta t) - 2(|\hat{\boldsymbol{\omega}}|\Delta t)\sin(|\hat{\boldsymbol{\omega}}|\Delta t) + 2}{2|\hat{\boldsymbol{\omega}}|^4}\lfloor\hat{\boldsymbol{\omega}}\rfloor^2)\Bigg)(\hat{\mathbf{a}}) \end{aligned} \tag{53}$$

The first two terms are from the previous integration time step, as we build these Jacobians incrementally. Defining $\hat{\mathbf{e}}_i$ as the unit vector in the i'th direction, and $\hat{\boldsymbol{\omega}}_i$ the corresponding entry in $\hat{\boldsymbol{\omega}}$ the third term can be found as the following:

$$\begin{aligned} \frac{\partial^k_{\tau+1}\mathbf{R}(\frac{(\Delta t)^2}{2}\mathbf{I} + f_1\lfloor\hat{\boldsymbol{\omega}}\rfloor + f_2\lfloor\hat{\boldsymbol{\omega}}\rfloor^2)}{\partial \mathbf{b}_{\omega_i}} = & \frac{\partial^k_{\tau+1}\mathbf{R}}{\partial \mathbf{b}_{\omega_i}}\left(\frac{(\Delta t)^2}{2}\mathbf{I} + f_1\lfloor\hat{\boldsymbol{\omega}}\rfloor + f_2\lfloor\hat{\boldsymbol{\omega}}\rfloor^2\right) \\ & + {}^k_{\tau+1}\breve{\mathbf{R}}\left(\frac{\partial f_1}{\partial \mathbf{b}_{\omega_i}}\lfloor\hat{\boldsymbol{\omega}}\rfloor - f_1\lfloor\hat{\mathbf{e}}_i\rfloor + \frac{\partial f_2}{\partial \mathbf{b}_{\omega_i}}\lfloor\hat{\boldsymbol{\omega}}\rfloor^2 - f_2(\lfloor\hat{\mathbf{e}}_i\rfloor\lfloor\hat{\boldsymbol{\omega}}\rfloor + \lfloor\hat{\boldsymbol{\omega}}\rfloor\lfloor\hat{\mathbf{e}}_i\rfloor)\right) \end{aligned} \tag{54}$$

where f_1 and f_2 are the corresponding coefficients in Equation (53), and their derivatives are computed as:

$$\frac{\partial f_1}{\partial \mathbf{b}_{\omega_i}} = \frac{\hat{\omega}_i(|\hat{\omega}|^2\Delta t^2 \sin(|\hat{\omega}|\Delta t) - 3\sin(|\hat{\omega}|\Delta t) + 3|\hat{\omega}|\Delta t \cos(|\hat{\omega}|\Delta t))}{|\hat{\omega}|^5} \tag{55}$$

$$\frac{\partial f_2}{\partial \mathbf{b}_{\omega_i}} = \frac{\hat{\omega}_i((|\hat{\omega}|\Delta t)^2 - 4\cos(|\hat{\omega}|\Delta t) - 4(|\hat{\omega}|\Delta t)\sin(|\hat{\omega}|\Delta t) + (|\hat{\omega}|\Delta t)^2\cos(|\hat{\omega}|\Delta t) + 4)}{|\hat{\omega}|^6} \tag{56}$$

For small $\hat{\omega}$,

$$\lim_{|\hat{\omega}|\to 0} \frac{\partial f_1}{\partial \mathbf{b}_{\omega_i}} = -\hat{\omega}_i \frac{\Delta t^5}{15} \tag{57}$$

$$\lim_{|\hat{\omega}|\to 0} \frac{\partial f_2}{\partial \mathbf{b}_{\omega_i}} = \hat{\omega}_i \frac{\Delta t^6}{72} \tag{58}$$

Similarly, we have:

$$\begin{aligned}\frac{\partial {}^k\boldsymbol{\beta}_{\tau+1}}{\partial \mathbf{b}_{\omega_i}} &= \frac{\partial {}^k\boldsymbol{\beta}_{\tau}}{\partial \mathbf{b}_{\omega_i}} + \frac{\partial^k_{\tau+1}\mathbf{R}}{\partial \mathbf{b}_{\omega_i}} \left(\Delta t \mathbf{I} + f_3\lfloor\hat{\omega}\rfloor + f_4\lfloor\hat{\omega}\rfloor^2\right)\hat{\mathbf{a}} \\ &+ {}^k_{\tau+1}\breve{\mathbf{R}}\left(\frac{\partial f_3}{\partial \mathbf{b}_{\omega_i}}\lfloor\hat{\omega}\rfloor - f_3\lfloor\hat{\mathbf{e}}_i\rfloor + \frac{\partial f_4}{\partial \mathbf{b}_{\omega_i}}\lfloor\hat{\omega}\rfloor^2 - f_4(\lfloor\hat{\mathbf{e}}_i\rfloor\lfloor\hat{\omega}\rfloor + \lfloor\hat{\omega}\rfloor\lfloor\hat{\mathbf{e}}_i\rfloor)\right)\hat{\mathbf{a}}\end{aligned} \tag{59}$$

where

$$\frac{\partial f_3}{\partial \mathbf{b}_{\omega_i}} = \frac{\hat{\omega}_i(2(\cos(|\hat{\omega}|\Delta t) - 1) + (|\hat{\omega}|\Delta t)\sin(|\hat{\omega}|\Delta t))}{|\hat{\omega}|^4} \tag{60}$$

$$\frac{\partial f_4}{\partial \mathbf{b}_{\omega_i}} = \frac{\hat{\omega}_i(2(|\hat{\omega}|\Delta t) + (|\hat{\omega}|\Delta t)\cos(|\hat{\omega}|\Delta t) - 3\sin(|\hat{\omega}|\Delta t))}{|\hat{\omega}|^5} \tag{61}$$

For small $\hat{\omega}$ we have:

$$\lim_{|\hat{\omega}|\to 0} \frac{\partial f_3}{\partial \mathbf{b}_{\omega_i}} = -\hat{\omega}_i \frac{\Delta t^4}{12} \tag{62}$$

$$\lim_{|\hat{\omega}|\to 0} \frac{\partial f_4}{\partial \mathbf{b}_{\omega_i}} = \hat{\omega}_i \frac{\Delta t^5}{60} \tag{63}$$

We now show how to derive the derivative of the rotation matrix with respect to a change in bias, i.e., $\Delta\mathbf{b}_\omega = \mathbf{b}_\omega - \mathbf{b}^\star_{\omega_k}$. To do so, we consider a set of measurements over the interval and for the first measurement interval, $[t_k, t_{\tau_1}]$, with measurement $\boldsymbol{\omega}_{m_1}$, we integrate over its sub-interval to get the following:

$$\begin{aligned}{}^{\tau_1}_k\mathbf{R} &= ({}^k_{\tau_1}\mathbf{R})^\top = (\exp(\lfloor\boldsymbol{\omega}_{m_1} - \mathbf{b}^\star_{\omega_k} - \Delta\mathbf{b}_\omega\rfloor\Delta t))^\top && (64)\\ &\simeq (\exp(\lfloor\boldsymbol{\omega}_{m_1} - \mathbf{b}^\star_{\omega_k}\rfloor\Delta t)\exp(\lfloor-\mathbf{J}_{r_1}\Delta\mathbf{b}_\omega\rfloor\Delta t))^\top && (65)\\ &= \exp(\lfloor\mathbf{J}_{r_1}\Delta\mathbf{b}_\omega\rfloor\Delta t)\exp(-\lfloor\boldsymbol{\omega}_{m_1} - \mathbf{b}^\star_{\omega_k}\rfloor\Delta t) && (66)\\ &= \exp(\lfloor\mathbf{J}_{r_1}\Delta\mathbf{b}_\omega\rfloor\Delta t){}^{\tau_1}_k\breve{\mathbf{R}} && (67)\\ &\simeq (\mathbf{I}_{3\times3} + \lfloor\mathbf{J}_{r_1}\Delta\mathbf{b}_\omega\rfloor\Delta t){}^{\tau_1}_k\breve{\mathbf{R}} && (68)\end{aligned}$$

where $\mathbf{J}_{r_i}$ is the right Jacobian of $SO(3)$ evaluated at the i'th measurement (i.e., $\boldsymbol{\omega}_{m_i}$), see Equation (164). This decomposition can be done for every measurement interval:

$$ {}^{\tau_{i+1}}_{\tau_i}\mathbf{R} \simeq (\mathbf{I}_{3\times3} + \lfloor \mathbf{J}_{r_{i+1}} \Delta\mathbf{b}_\omega \rfloor \Delta t) {}^{\tau_{i+1}}_{\tau_i}\breve{\mathbf{R}} \tag{69} $$

We can now look at what happens when we compound measurements. In particular, at the second-step $[t_{\tau_1}, t_{\tau_2}]$, we have the following:

$$ \begin{aligned} {}^{\tau_2}_{k}\mathbf{R} &= {}^{\tau_2}_{\tau_1}\mathbf{R}\,{}^{\tau_1}_{k}\mathbf{R} && (70)\\ &\simeq (\mathbf{I}_{3\times3} + \lfloor \mathbf{J}_{r_2}\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_2}_{\tau_1}\breve{\mathbf{R}}(\mathbf{I}_{3\times3} + \lfloor \mathbf{J}_{r_1}\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_1}_{k}\breve{\mathbf{R}} && (71)\\ &= (\mathbf{I}_{3\times3} + \lfloor \mathbf{J}_{r_2}\Delta\mathbf{b}_\omega \rfloor \Delta t)(\mathbf{I}_{3\times3} + \lfloor {}^{\tau_2}_{\tau_1}\breve{\mathbf{R}}\mathbf{J}_{r_1}\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_2}_{k}\breve{\mathbf{R}} && (72)\\ &\approx (\mathbf{I}_{3\times3} + \lfloor (\mathbf{J}_{r_2} + {}^{\tau_2}_{\tau_1}\breve{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_2}_{k}\breve{\mathbf{R}} && (73) \end{aligned} $$

Here we have used the property that $\mathbf{R}\lfloor \mathbf{w} \rfloor \mathbf{R}^\top = \lfloor \mathbf{R}\mathbf{w} \rfloor$ for a rotation matrix. Repeating this process for the interval $[t_{\tau_2}, t_{\tau_3}]$ yields:

$$ \begin{aligned} {}^{\tau_3}_{k}\mathbf{R} &= {}^{\tau_3}_{\tau_2}\mathbf{R}\,{}^{\tau_2}_{k}\mathbf{R} && (74)\\ &\simeq (\mathbf{I}_{3\times3} + \lfloor \mathbf{J}_{r_3}\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_3}_{\tau_2}\breve{\mathbf{R}}(\mathbf{I}_{3\times3} + \lfloor (\mathbf{J}_{r_2} + {}^{\tau_2}_{\tau_1}\breve{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_2}_{k}\breve{\mathbf{R}} && (75)\\ &= (\mathbf{I}_{3\times3} + \lfloor \mathbf{J}_{r_3}\Delta\mathbf{b}_\omega \rfloor)(\mathbf{I}_{3\times3} + \lfloor ({}^{\tau_3}_{\tau_2}\breve{\mathbf{R}}\mathbf{J}_{r_2} + {}^{\tau_3}_{\tau_1}\breve{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_3}_{k}\breve{\mathbf{R}} && (76)\\ &\simeq (\mathbf{I}_{3\times3} + \lfloor (\mathbf{J}_{r_3} + {}^{\tau_3}_{\tau_2}\breve{\mathbf{R}}\mathbf{J}_{r_2} + {}^{\tau_3}_{\tau_1}\breve{\mathbf{R}}\mathbf{J}_{r_1})\Delta\mathbf{b}_\omega \rfloor \Delta t){}^{\tau_3}_{k}\breve{\mathbf{R}} && (77) \end{aligned} $$

We thus see the pattern developing and can write the updated rotation at any time step t_u as:

$$ \begin{aligned} {}^{u}_{k}\mathbf{R} &= \exp(\lfloor \mathbf{J}_q(u)(\mathbf{b}_\omega - \mathbf{b}^\star_{\omega_k}) \rfloor){}^{u}_{k}\breve{\mathbf{R}} && (78)\\ \text{with} \;\; \mathbf{J}_q(u) &= \sum_{\tau=\tau_1}^{u} {}^{u}_{\tau}\breve{\mathbf{R}}\mathbf{J}_{r_\tau}\Delta t && (79) \end{aligned} $$

Each of these values can be calculated *incrementally* by noting that:

$$ \begin{aligned} \mathbf{J}_q(u+1) &= {}^{u+1}_{u}\breve{\mathbf{R}} \sum_{\tau=\tau_1}^{u} {}^{u}_{\tau}\breve{\mathbf{R}}\mathbf{J}_{r_\tau}\Delta t + \mathbf{J}_{r_{u+1}}\Delta t && (80)\\ &= {}^{u+1}_{u}\breve{\mathbf{R}}\mathbf{J}_q(u) + \mathbf{J}_{r_{u+1}}\Delta t && (81) \end{aligned} $$

The derivative of every rotation with respect to the i'th entry of the gyro bias, which appears in both Equation (54) and (59) can be approximated using:

$$ \begin{aligned} {}^{u}_{k}\mathbf{R} &\simeq (\mathbf{I}_{3\times3} + \lfloor \mathbf{J}_q(u)(\mathbf{b}_\omega - \mathbf{b}^\star_{\omega_k}) \rfloor){}^{u}_{k}\breve{\mathbf{R}} && (82)\\ \frac{\partial {}^{u}_{k}\mathbf{R}}{\partial \mathbf{b}_{\omega_i}} &\approx \lfloor \mathbf{J}_q(u)\hat{\mathbf{e}}_i \rfloor {}^{u}_{k}\breve{\mathbf{R}} && (83)\\ \frac{\partial {}^{k}_{u}\mathbf{R}}{\partial \mathbf{b}_{\omega_i}} &\approx -{}^{k}_{u}\breve{\mathbf{R}}\lfloor \mathbf{J}_q(u)\hat{\mathbf{e}}_i \rfloor && (84) \end{aligned} $$

The total rotation after a bias update can be expressed as:

$$ {}^{k+1}_{k}\mathbf{R} = \exp(\lfloor \mathbf{J}_q(k+1)(\mathbf{b}_\omega - \mathbf{b}^\star_{\omega_k}) \rfloor){}^{k+1}_{k}\breve{\mathbf{R}} \tag{85} $$

$$ \tag{86} $$

4.6 Measurement Residual

$$\mathbf{e}_{IMU}(\mathbf{x}) = \tag{87}$$

$$\begin{bmatrix} 2\mathbf{vec}\left({}^{k+1}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \bar{q}_b\right) \\ \hdashline \mathbf{b}_{\omega_{k+1}} - \mathbf{b}_{\omega_k} \\ \hdashline \left({}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_k + {}^{G}\mathbf{g}\Delta T\right) - \mathbf{J}_\beta\left(\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k}\right) - \mathbf{H}_\beta\left(\mathbf{b}_{a_k} - \mathbf{b}^\star_{a_k}\right) - {}^{k}\breve{\boldsymbol{\beta}}_{k+1}\right) \\ \hdashline \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \hdashline \left({}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_k - {}^{G}\mathbf{v}_k\Delta T + \frac{1}{2}{}^{G}\mathbf{g}\Delta T^2\right) - \mathbf{J}_\alpha\left(\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k}\right) - \mathbf{H}_\alpha\left(\mathbf{b}_{a_k} - \mathbf{b}^\star_{a_k}\right) - {}^{k}\breve{\boldsymbol{\alpha}}_{k+1}\right) \end{bmatrix}$$

where

$$\bar{q}_b = \begin{bmatrix} \frac{\boldsymbol{\theta}}{||\boldsymbol{\theta}||} \sin\left(\frac{||\boldsymbol{\theta}||}{2}\right) \\ \cos\left(\frac{||\boldsymbol{\theta}||}{2}\right) \end{bmatrix} \tag{88}$$

$$\boldsymbol{\theta} = \mathbf{J}_q\left(\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k}\right) \tag{89}$$

4.7 Measurement Jacobian

Let us partition the preintegrated residual for the first model as:

$$\mathbf{e}_{IMU} = \begin{bmatrix} \mathbf{e}_\theta^\top & \mathbf{e}_{b_\omega}^\top & \mathbf{e}_v^\top & \mathbf{e}_{b_a}^\top & \mathbf{e}_p^\top \end{bmatrix}^\top \tag{90}$$

The measurement Jacobian with respect to one element of the state vector can be found by perturbing the measurement function by the corresponding element. For example, the relative-rotation measurement residual is perturbed by a change in gyro bias around the current estimate (i.e., $\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k} = \hat{\mathbf{b}}_{\omega_k} + \delta\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k}$):

$$\begin{aligned} \mathbf{e}_\theta &= 2\mathbf{vec}\left({}^{k+1}_{G}\hat{\bar{q}} \otimes {}^{k}_{G}\hat{\bar{q}}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_{\omega_k} + \delta\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k})}{2} \\ 1 \end{bmatrix}\right) \\ &=: 2\mathbf{vec}\left(\hat{\bar{q}}_r \otimes \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_{\omega_k} + \delta\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k})}{2} \\ 1 \end{bmatrix}\right) \\ &= 2\mathbf{vec}\left(\mathcal{L}(\hat{\bar{q}}_r) \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_{\omega_k} + \delta\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k})}{2} \\ 1 \end{bmatrix}\right) \\ &= 2\mathbf{vec}\left(\begin{bmatrix} \hat{q}_{r,4}\mathbf{I}_{3\times3} - \lfloor\hat{\mathbf{q}}_r\rfloor & \hat{\mathbf{q}}_r \\ -\hat{\mathbf{q}}_r^\top & \hat{q}_{r,4} \end{bmatrix} \begin{bmatrix} \frac{\mathbf{J}_q(\hat{\mathbf{b}}_{\omega_k} + \delta\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k})}{2} \\ 1 \end{bmatrix}\right) \\ &= (\hat{q}_{r,4}\mathbf{I}_{3\times3} - \lfloor\hat{\mathbf{q}}_r\rfloor)\mathbf{J}_q(\hat{\mathbf{b}}_{\omega_k} + \delta\mathbf{b}_{\omega_k} - \mathbf{b}^\star_{\omega_k}) + \text{other terms} \end{aligned}$$

So that our Jacobian with respect to a perturbance in bias is:

$$\frac{\partial\mathbf{e}_\theta}{\partial\delta\mathbf{b}_{\omega_k}} = (\hat{q}_{r,4}\mathbf{I}_{3\times3} - \lfloor\hat{\mathbf{q}}_r\rfloor)\mathbf{J}_q \tag{91}$$

Similarly, the Jacobian with respect to ${}^{k+1}\delta\boldsymbol{\theta}_G$ can be found as follows:

$$
\begin{aligned}
\mathbf{e}_\theta &= 2\mathbf{vec}\left(\begin{bmatrix}\frac{{}^{k+1}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\otimes {}^{k+1}_{G}\hat{\bar{q}}\otimes {}^{k}_{G}\hat{\bar{q}}^{-1}\otimes {}^{k+1}_{k}\bar{q}^{-1}\otimes\hat{\bar{q}}_b\right)\\
&= 2\mathbf{vec}\left(\begin{bmatrix}\frac{{}^{k+1}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\otimes\hat{\bar{q}}_{rb}\right)\\
&= 2\mathbf{vec}\left(\mathcal{R}(\hat{q}_{rb})\begin{bmatrix}\frac{{}^{k+1}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\right)\\
&= 2\mathbf{vec}\left(\begin{bmatrix}\hat{q}_{rb,4}\mathbf{I}_{3\times 3}+\lfloor\hat{\mathbf{q}}_{rb}\rfloor & \hat{\mathbf{q}}_{rb}\\ -\hat{\mathbf{q}}_{rb}^\top & \hat{q}_{rb,4}\end{bmatrix}\begin{bmatrix}\frac{{}^{k+1}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\right)\\
&= (\hat{q}_{rb,4}\mathbf{I}_{3\times 3}+\lfloor\hat{\mathbf{q}}_{rb}\rfloor)^{k+1}\delta\boldsymbol{\theta}_G+\text{ other terms}
\end{aligned}
$$

Yielding the Jacobian:

$$
\frac{\partial\mathbf{e}_\theta}{\partial^{k+1}\delta\boldsymbol{\theta}_G}=\hat{q}_{rb,4}\mathbf{I}_{3\times 3}+\lfloor\hat{\mathbf{q}}_{rb}\rfloor \tag{92}
$$

The Jacobian with respect to ${}^{k}\delta\boldsymbol{\theta}_G$ is given by:

$$
\begin{aligned}
\mathbf{e}_\theta &= 2\mathbf{vec}\left({}^{k+1}_{G}\hat{\bar{q}}\otimes {}^{k}_{G}\hat{\bar{q}}^{-1}\otimes\begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\otimes {}^{k+1}_{k}\bar{q}^{-1}\otimes\hat{\bar{q}}_b\right)\\
&= 2\mathbf{vec}\left(\hat{q}_n\otimes\begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\otimes\hat{q}_{mb}^{-1}\right)\\
&= 2\mathbf{vec}\left(\mathcal{L}(\hat{q}_n)\mathcal{R}(\bar{q}_{mb}^{-1})\begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\right)\\
&= 2\mathbf{vec}\Big(\begin{bmatrix}\hat{q}_{n,4}\mathbf{I}_{3\times 3}-\lfloor\hat{\mathbf{q}}_n\rfloor & \hat{\mathbf{q}}_n\\ -\hat{\mathbf{q}}_n^\top & \hat{q}_{n,4}\end{bmatrix}\\
&\quad\begin{bmatrix}\bar{q}_{mb,4}\mathbf{I}_{3\times 3}-\lfloor\bar{\mathbf{q}}_{mb}\rfloor & -\mathbf{q}_{mb}\\ \mathbf{q}_{mb}^\top & \bar{q}_{mb,4}\end{bmatrix}\begin{bmatrix}-\frac{{}^{k}\delta\boldsymbol{\theta}_G}{2}\\ 1\end{bmatrix}\Big)\\
&= -((\hat{q}_{n,4}\mathbf{I}_{3\times 3}-\lfloor\hat{\mathbf{q}}_n\rfloor)(q_{mb,4}\mathbf{I}_{3\times 3}-\lfloor\mathbf{q}_{mb}\rfloor)+\hat{\mathbf{q}}_n\mathbf{q}_{mb}^\top)^{k}\delta\boldsymbol{\theta}_G+\text{ other terms}
\end{aligned}
$$

Which gives the Jacobian:

$$
\frac{\partial\mathbf{e}_\theta}{\partial^{k}\delta\boldsymbol{\theta}_G}=-((\hat{q}_{n,4}\mathbf{I}_{3\times 3}-\lfloor\hat{\mathbf{q}}_n\rfloor)(\bar{q}_{mb,4}\mathbf{I}_{3\times 3}-\lfloor\mathbf{q}_{mb}\rfloor)+\hat{\mathbf{q}}_n\bar{\mathbf{q}}_{mb}^\top)
$$

Note than in the preceding Jacobians, we have defined several intermediate quaternions, ($\hat{\bar{q}}_r, \hat{\bar{q}}_{rb}, \hat{\bar{q}}_n$, and $\hat{\bar{q}}_{mb}$) for ease of notation. Following the same methodology, we can find the Jacobians of the $\boldsymbol{\alpha}$ measurement with respect to the position, velocity and bias.

$$
\frac{\partial\mathbf{e}_{b_\omega}}{\partial\delta\mathbf{b}_{\omega_k}}=-\mathbf{I} \tag{93}
$$

$$
\frac{\partial\mathbf{e}_{b_\omega}}{\partial\delta\mathbf{b}_{\omega_{k+1}}}=\mathbf{I} \tag{94}
$$

$$\frac{\partial \mathbf{e}_v}{\partial^k \delta \boldsymbol{\theta}_G} = \left\lfloor {}^k_G\hat{\mathbf{R}}({}^G\hat{\mathbf{v}}_{k+1} - {}^G\hat{\mathbf{v}}_k + {}^G\mathbf{g}\Delta t) \right\rfloor \tag{95}$$

$$\frac{\partial \mathbf{e}_v}{\partial \delta \mathbf{b}_{\omega_k}} = -\mathbf{J}_\beta \tag{96}$$

$$\frac{\partial \mathbf{e}_v}{\partial^G \delta \mathbf{v}_k} = -{}^k_G\hat{\mathbf{R}} \tag{97}$$

$$\frac{\partial \mathbf{e}_v}{\partial^G \delta \mathbf{v}_{k+1}} = {}^k_G\hat{\mathbf{R}} \tag{98}$$

$$\frac{\partial \mathbf{e}_v}{\partial \delta \mathbf{b}_a} = -\mathbf{H}_\beta \tag{99}$$

$$\frac{\partial \mathbf{e}_{b_a}}{\partial \delta \mathbf{b}_{a_k}} = -\mathbf{I} \tag{100}$$

$$\frac{\partial \mathbf{e}_{b_a}}{\partial \delta \mathbf{b}_{a_{k+1}}} = \mathbf{I} \tag{101}$$

$$\frac{\partial \mathbf{e}_p}{\partial^k \delta \boldsymbol{\theta}_G} = \left\lfloor {}^k_G\hat{\mathbf{R}} \left({}^G\hat{\mathbf{p}}_{k+1} - {}^G\hat{\mathbf{p}}_k - {}^G\hat{\mathbf{v}}_k \Delta t + \frac{1}{2} {}^G\mathbf{g}\Delta t^2 \right) \right\rfloor \tag{102}$$

$$\frac{\partial \mathbf{e}_p}{\partial \delta \mathbf{b}_{\omega_k}} = -\mathbf{J}_\alpha \tag{103}$$

$$\frac{\partial \mathbf{e}_p}{\partial^G \delta \mathbf{v}_k} = -{}^k_G\hat{\mathbf{R}}\Delta t \tag{104}$$

$$\frac{\partial \mathbf{e}_p}{\partial \delta \mathbf{b}_{a_k}} = -\mathbf{H}_\alpha \tag{105}$$

$$\frac{\partial \mathbf{e}_p}{\partial^G \delta \mathbf{p}_k} = -{}^k_G\hat{\mathbf{R}} \tag{106}$$

$$\frac{\partial \mathbf{e}_p}{\partial^G \delta \mathbf{p}_{k+1}} = {}^k_G\hat{\mathbf{R}} \tag{107}$$

$$\tag{108}$$

5 Model 2: Piecewise Constant Local Acceleration

The previous preintegration (Model 1) assumes that noiseless IMU measurements can be approximated as remaining constant over a sampling interval, which, however, might not always be a good approximation. For example, in the case of an IMU rotating against the direction of gravity, the measurement will change over a sampling interval continuously due to the effect of gravity. In this section, we propose a new preintegration model that instead assumes piecewise constant *true* local acceleration during the sampling time interval, which may better approximate motion dynamics in practice.

To this end, we first rewrite the state dynamics as:

$${}^G\mathbf{p}_{k+1} = {}^G\mathbf{p}_k + {}^G\mathbf{v}_k \Delta T + {}^G_k\mathbf{R} \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} {}^k_u\mathbf{R}\mathbf{a} \, du ds \tag{109}$$

$${}^G\mathbf{v}_{k+1} = {}^G\mathbf{v}_k + {}^G_k\mathbf{R} \int_{t_k}^{t_{k+1}} {}^k_u\mathbf{R}\mathbf{a} \, du \tag{110}$$

Note that we have moved the effect of gravity back inside the integrals. We then define the following vectors:

$$\Delta p = \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} {}^{k}_{u}\mathbf{R}\mathbf{a} \; duds \tag{111}$$

$$\Delta v = \int_{t_k}^{t_{k+1}} {}^{k}_{u}\mathbf{R}\mathbf{a} \; du \tag{112}$$

which essentially are the true local position displacement and velocity change during $[t_k, t_{k+1}]$, and yields:

$$\Delta \dot{p} = \Delta v \tag{113}$$

$$\Delta \dot{v} = {}^{k}_{u}\mathbf{R}\mathbf{a} \tag{114}$$

In particular, between two IMU measurement times inside the preintegration interval, $[t_\tau, t_{\tau+1}] \subset [t_k, t_{k+1}]$, we assume that the *local* acceleration will be constant:

$$\forall t_u \in [t_\tau, t_{\tau+1}], \qquad \mathbf{a}(t_u) = \mathbf{a}(t_\tau) \tag{115}$$

Using this sampling model we can rewrite Equation (114) as:

$$\Delta \dot{v} = {}^{k}_{u}\mathbf{R}\left(\mathbf{a}_m - \mathbf{b}_a - \mathbf{n}_a - {}^{\tau}_{k}\mathbf{R}{}^{k}_{G}\mathbf{R}{}^{G}\mathbf{g}\right) \tag{116}$$

We now write the relationship of the states at the beginning and end of the interval as (see Equations (109) and (110)):

$${}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_{k} - {}^{G}\mathbf{v}_{k}\Delta T\right) = \Delta p \tag{117}$$

$${}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_{k}\right) = \Delta v \tag{118}$$

It is important to note that, since Δp and Δv are functions of both the biases *and* the initial orientation, we perform the following linearization with respect to these states:

$$\begin{aligned}{}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_{k} - {}^{G}\mathbf{v}_{k}\Delta T\right) \simeq \Delta p\left(\mathbf{b}^{\star}_{\omega_k}, \mathbf{b}^{\star}_{a_k}, {}^{k}_{G}\bar{q}^{\star}\right) \\ + \left.\frac{\partial \Delta p}{\partial \mathbf{b}_\omega}\right|_{\mathbf{b}^{\star}_{\omega_k}} \Delta \mathbf{b}_\omega + \left.\frac{\partial \Delta p}{\partial \mathbf{b}_a}\right|_{\mathbf{b}^{\star}_{a_k}} \Delta \mathbf{b}_a + \left.\frac{\partial \Delta p}{\partial \Delta \boldsymbol{\theta}_k}\right|_{{}^{k}_{G}\bar{q}^{\star}} \Delta \boldsymbol{\theta}_k\end{aligned} \tag{119}$$

$${}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_{k}\right) \simeq \Delta v\left(\mathbf{b}^{\star}_{\omega_k}, \mathbf{b}^{\star}_{a_k}, {}^{k}_{G}\bar{q}^{\star}\right) + \left.\frac{\partial \Delta v}{\partial \mathbf{b}_\omega}\right|_{\mathbf{b}^{\star}_{\omega_k}} \Delta \mathbf{b}_\omega + \left.\frac{\partial \Delta v}{\partial \mathbf{b}_a}\right|_{\mathbf{b}^{\star}_{a_k}} \Delta \mathbf{b}_a + \left.\frac{\partial \Delta v}{\partial \Delta \boldsymbol{\theta}_k}\right|_{{}^{k}_{G}\bar{q}^{\star}} \Delta \boldsymbol{\theta}_k \tag{120}$$

where $\Delta \boldsymbol{\theta}_k = 2\mathbf{vec}\left({}^{k}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{\star-1}\right)$ is the rotation angle change associated with the change of the linearization point of quaternion ${}^{k}_{G}\bar{q}$.

5.1 Measurement Mean

To compute the new preintegrated measurement mean values, we first determine the continuous-time dynamics of the expected preintegration vectors by taking expectations of Equations (113) and (116), given by:

$$\Delta \dot{\breve{p}} = \Delta \breve{v} \tag{121}$$

$$\Delta \dot{\breve{v}} = {}^{k}_{u}\breve{\mathbf{R}}\left(\mathbf{a}_m - \mathbf{b}^{\star}_{a_k} - {}^{\tau}_{k}\breve{\mathbf{R}}{}^{k}_{G}\mathbf{R}^{\star G}\mathbf{g}\right) \tag{122}$$

As in the case of Model 1, we can formulate a linear system of the new preintegration measurement vectors and find the closed-from solutions. Specifically, we can integrate these differential equations and obtain the solution similar to Equation (37), while using the new definition: $\hat{\mathbf{a}} = \mathbf{a}_m - \mathbf{b}^{\star}_{a_k} - {}^{\tau}_{k}\breve{\mathbf{R}}{}^{k}_{G}\mathbf{R}^{\star G}\mathbf{g}$, which serves as the estimate for the piecewise constant local acceleration over the sampling interval.

5.2 Measurement Covariance

In order to use the derived continuous preintegration measurement means, we must have the covariances associated with their error quantities. We compute the derivative of our measurement error states with respect to error sources as follows:

$$\Delta\dot{\tilde{p}} = \Delta v - \Delta\breve{v} \tag{123}$$

$$= \Delta\tilde{v} \tag{124}$$

$$\Delta\dot{\tilde{v}} = {}^{k}_{u}\breve{\mathbf{R}}\left(\mathbf{I} + \lfloor {}^{u}\delta\boldsymbol{\theta}_k \rfloor\right)\left(\mathbf{a}_m - \mathbf{b}^{\star}_{a_k} - \tilde{\mathbf{b}}_a - \left(\mathbf{I} - \lfloor {}^{\tau}\delta\boldsymbol{\theta}_k \rfloor\right){}^{\tau}_{k}\breve{\mathbf{R}}\left(\mathbf{I} - \lfloor \Delta\boldsymbol{\theta}_k \rfloor\right){}^{k}_{G}\mathbf{R}^{\star G}\mathbf{g} - \mathbf{n}_a\right) - {}^{k}_{u}\breve{\mathbf{R}}\left(\mathbf{a}_m - \mathbf{b}^{\star}_{a_k} - {}^{\tau}_{k}\breve{\mathbf{R}}{}^{k}_{G}\mathbf{R}^{\star G}\mathbf{g}\right) \tag{125}$$

$$= -{}^{k}_{u}\breve{\mathbf{R}}\lfloor \hat{\mathbf{a}} \rfloor {}^{u}\delta\boldsymbol{\theta}_k - {}^{k}_{u}\breve{\mathbf{R}}\tilde{\mathbf{b}}_a - {}^{k}_{u}\breve{\mathbf{R}}\lfloor {}^{\tau}\breve{\mathbf{g}} \rfloor {}^{\tau}\delta\boldsymbol{\theta}_k - {}^{k}_{u}\breve{\mathbf{R}}{}^{\tau}_{k}\breve{\mathbf{R}}\lfloor {}^{k}\mathbf{g}^{\star} \rfloor \Delta\boldsymbol{\theta}_k - {}^{k}_{u}\breve{\mathbf{R}}\mathbf{n}_a \tag{126}$$

where ${}^{k}\mathbf{g}^{\star} = {}^{k}_{G}\mathbf{R}^{\star G}\mathbf{g}$ is the global gravity rotated into the local frame of the linearization point. In the above expressions, we have used three angle errors: (i) ${}^{u}\delta\boldsymbol{\theta}_k$ corresponds to the active local IMU orientation error, (ii) ${}^{\tau}\delta\boldsymbol{\theta}_k$ corresponds to the cloned orientation error at the sampling time t_τ, and (iii) $\Delta\boldsymbol{\theta}_k$ is the *global* angle error of the starting orientation of the preintegration time interval.

In addition, the bias errors $\tilde{\mathbf{b}}$ describe the deviation of the bias from the starting value over the interval due to bias drift. With this, we have the following time evolution of the full preintegrated measurement error state:

$$\begin{bmatrix} {}^{u}\delta\dot{\boldsymbol{\theta}}_k \\ \dot{\tilde{\mathbf{b}}}_\omega \\ \Delta\dot{\tilde{v}} \\ \dot{\tilde{\mathbf{b}}}_a \\ \Delta\dot{\tilde{p}} \\ {}^{\tau}\delta\dot{\boldsymbol{\theta}}_k \\ \Delta\dot{\boldsymbol{\theta}}_k \end{bmatrix} = \mathbf{F}\begin{bmatrix} {}^{u}\delta\boldsymbol{\theta}_k \\ \tilde{\mathbf{b}}_\omega \\ \Delta\tilde{v} \\ \tilde{\mathbf{b}}_a \\ \Delta\tilde{p} \\ {}^{\tau}\delta\boldsymbol{\theta}_k \\ \Delta\boldsymbol{\theta}_k \end{bmatrix} + \begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -{}^{k}_{u}\breve{\mathbf{R}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{n}_\omega \\ \mathbf{n}_{\omega b} \\ \mathbf{n}_a \\ \mathbf{n}_{ab} \end{bmatrix} \tag{127}$$

$$\Rightarrow \quad \dot{\mathbf{r}} = \mathbf{F}\mathbf{r} + \mathbf{G}\mathbf{n} \tag{128}$$

where

$$\mathbf{F} = \begin{bmatrix} -\lfloor \boldsymbol{\omega} \rfloor & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -{}^{k}_{u}\breve{\mathbf{R}}\lfloor \hat{\mathbf{a}} \rfloor & \mathbf{0} & \mathbf{0} & -{}^{k}_{u}\breve{\mathbf{R}} & \mathbf{0} & -{}^{k}_{u}\breve{\mathbf{R}}\lfloor {}^{\tau}\breve{\mathbf{g}} \rfloor & -{}^{k}_{u}\breve{\mathbf{R}}{}^{\tau}_{k}\breve{\mathbf{R}}\lfloor {}^{k}\mathbf{g}^{\star} \rfloor \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{129}$$

The above continuous time measurement error evolution can be can be numerically integrated to get the appropriate state transition $\boldsymbol{\Phi}(t_{\tau+1}, t_\tau)$ and additive measurement noise covariance $\mathbf{Q}_\tau$.

Having defined the state transition matrix and additive measurement noise covariance, for each incoming IMU measurement we can "propagate" our total preintegration covariance $\mathbf{P}$ as follows:

$$\mathbf{P}_k = \mathbf{0}_{21\times 21} \tag{130}$$

$$\mathbf{P}^-_{\tau+1} = \mathbf{\Phi}(t_{\tau+1}, t_\tau)\mathbf{P}_\tau \mathbf{\Phi}(t_{\tau+1}, t_\tau)^\top + \mathbf{Q}_\tau \tag{131}$$

$$\mathbf{P}_{\tau+1} = \mathbf{B}\mathbf{P}^-_{\tau+1}\mathbf{B}^\top \tag{132}$$

where $\mathbf{B}$ is the cloning matrix that allows us to replace the previous static orientation error ${}^{\tau}\delta\boldsymbol{\theta}_k$ to the new one ${}^{\tau+1}\delta\boldsymbol{\theta}_k$ when moving to the next preintegration measurement time interval (i.e., from $[t_\tau, t_{\tau+1}]$ to $[t_{\tau+1}, t_{\tau+2}]$), and is given by:

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{133}$$

The resulting preintegrated measurement covariance is extracted from the top left 15×15 block of $\mathbf{P}_{k+1}$ after the propagation in Equations (130)-(132) is over for the entire preintegration interval $[t_k, t_{k+1}]$.

5.3 Bias Jacobian Discussion

Model two preintegration depends on the initial orientation ${}^{\tau}_{k}\mathbf{R}$ for each IMU sampling interval $[t_\tau, t_{\tau+1}]$ and thus, when integrating over one interval to the other, this initial orientation error changes from ${}^{\tau}\delta\boldsymbol{\theta}_k$ to ${}^{\tau+1}\delta\boldsymbol{\theta}_k$. This requires to extra care when computing the total Jacobian (state transition) matrix, $\mathbf{\Psi}_{k+1}$, over the entire preintegration interval $[t_k, t_{k+1}]$; that is, we propagate this matrix as follows:

$$\mathbf{\Psi}_k = \mathbf{I}_{21\times 21} \tag{134}$$

$$\mathbf{\Psi}_{\tau+1} = \mathbf{B}\mathbf{\Phi}(t_{\tau+1}, t_\tau)\mathbf{\Psi}_\tau \tag{135}$$

where $\mathbf{B}$ is defined the same as in Equation (133).

Clearly, the total Jacobin matrix $\mathbf{\Psi}_{k+1}$ will be the Jacobian of the resulting preintegrated measurement error with respect to the initial error. Therefore, we can simply extract the corresponding blocks to obtain the bias and initial orientation Jacobians. Denoting $\mathbf{\Psi}_{k+1}(i,j)$ the 3×3 block of the Jacobian matrix starting at index (i,j), we have:

$$\mathbf{J}_q = -\mathbf{\Psi}_{k+1}(0,3) \tag{136}$$

$$\mathbf{J}_a = \mathbf{\Psi}_{k+1}(12,3) \tag{137}$$

$$\mathbf{J}_b = \mathbf{\Psi}_{k+1}(6,3) \tag{138}$$

$$\mathbf{H}_a = \mathbf{\Psi}_{k+1}(12,9) \tag{139}$$

$$\mathbf{H}_b = \mathbf{\Psi}_{k+1}(6,9) \tag{140}$$

$$\mathbf{O}_a = \mathbf{\Psi}_{k+1}(12,18) \tag{141}$$

$$\mathbf{O}_b = \mathbf{\Psi}_{k+1}(6,18) \tag{142}$$

where we flip the sign of $\mathbf{J}_q$ to match the same definition used in Model 1.

5.4 Measurement Residual

$$\mathbf{e}_{IMU}(\mathbf{x}) = \begin{bmatrix} 2\mathbf{vec}\left({}^{k+1}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{-1} \otimes {}^{k+1}_{k}\breve{\bar{q}}^{-1} \otimes \bar{q}_b\right) \\ \hdashline \mathbf{b}_{\omega_{k+1}} - \mathbf{b}_{\omega_k} \\ \hdashline \Big({}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{v}_{k+1} - {}^{G}\mathbf{v}_k\right) - \mathbf{J}_\beta\left(\mathbf{b}_{\omega_k} - \mathbf{b}^{\star}_{\omega_k}\right) - \\ \mathbf{H}_\beta\left(\mathbf{b}_{a_k} - \mathbf{b}^{\star}_{a_k}\right) - \mathbf{O}_\beta\ 2\mathbf{vec}\left({}^{k}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{\star -1}\right) - \Delta\breve{v}\Big) \\ \hdashline \mathbf{b}_{a_{k+1}} - \mathbf{b}_{a_k} \\ \hdashline \Big({}^{k}_{G}\mathbf{R}\left({}^{G}\mathbf{p}_{k+1} - {}^{G}\mathbf{p}_k - {}^{G}\mathbf{v}_k\Delta T\right) - \mathbf{J}_\alpha\left(\mathbf{b}_{\omega_k} - \mathbf{b}^{\star}_{\omega_k}\right) - \\ \mathbf{H}_\alpha\left(\mathbf{b}_{a_k} - \mathbf{b}^{\star}_{a_k}\right) - \mathbf{O}_\alpha\ 2\mathbf{vec}\left({}^{k}_{G}\bar{q} \otimes {}^{k}_{G}\bar{q}^{\star -1}\right) - \Delta\breve{p}\Big) \end{bmatrix} \tag{143}$$

where

$$\bar{q}_b = \begin{bmatrix} \frac{\boldsymbol{\theta}}{||\boldsymbol{\theta}||}\ \sin\left(\frac{||\boldsymbol{\theta}||}{2}\right) \\ \cos\left(\frac{||\boldsymbol{\theta}||}{2}\right) \end{bmatrix} \tag{144}$$

$$\boldsymbol{\theta} = \mathbf{J}_q\left(\mathbf{b}_{\omega_k} - \mathbf{b}^{\star}_{\omega_k}\right) \tag{145}$$

5.5 Measurement Jacobian

Let us partition the preintegrated residual for the second model as:

$$\mathbf{e}_{IMU} = \begin{bmatrix} \mathbf{e}_\theta^\top & \mathbf{e}_{b_\omega}^\top & \mathbf{e}_v^\top & \mathbf{e}_{b_a}^\top & \mathbf{e}_p^\top \end{bmatrix}^\top \tag{146}$$

Instead of directly computing the derivatives, the measurement Jacobian with respect to one element of the state vector can be found by perturbing the measurement function by the corresponding element. The orientation and bias change measurement Jacobians remain unchanged under the new model. However, since the definition of $\mathbf{e}_v$ and $\mathbf{e}_p$ have changed, their Jacobians must be changed appropriately. The Jacobians are as follows:

$$\frac{\partial \mathbf{e}_{b_\omega}}{\partial \delta \mathbf{b}_{\omega_k}} = -\mathbf{I} \tag{147}$$

$$\frac{\partial \mathbf{e}_{b_\omega}}{\partial \delta \mathbf{b}_{\omega_{k+1}}} = \mathbf{I} \tag{148}$$

$$\frac{\partial \mathbf{e}_v}{\partial {}^{k}\delta\boldsymbol{\theta}_G} = \lfloor {}^{k}_{G}\hat{\mathbf{R}}\left({}^{G}\hat{\mathbf{v}}_{k+1} - {}^{G}\hat{\mathbf{v}}_k\right)\rfloor - \mathbf{O}_\beta\left(\tilde{q}_4\mathbf{I} + \lfloor\tilde{\mathbf{q}}\rfloor\right) \tag{149}$$

$$\frac{\partial \mathbf{e}_v}{\partial \delta \mathbf{b}_{\omega_k}} = -\mathbf{J}_\beta \tag{150}$$

$$\frac{\partial \mathbf{e}_v}{\partial {}^{G}\delta \mathbf{v}_k} = -{}^{k}_{G}\hat{\mathbf{R}} \tag{151}$$

$$\frac{\partial \mathbf{e}_v}{\partial^G \delta \mathbf{v}_{k+1}} = {}^k_G\hat{\mathbf{R}} \tag{152}$$

$$\frac{\partial \mathbf{e}_v}{\partial \delta \mathbf{b}_{a_k}} = -\mathbf{H}_\beta \tag{153}$$

$$\frac{\partial \mathbf{e}_{b_a}}{\partial \delta \mathbf{b}_{a_{k+1}}} = \mathbf{I} \tag{154}$$

$$\frac{\partial \mathbf{e}_{b_a}}{\partial \delta \mathbf{b}_{a_k}} = -\mathbf{I} \tag{155}$$

$$\frac{\partial \mathbf{e}_p}{\partial^k \delta \boldsymbol{\theta}_G} = \lfloor {}^k_G\hat{\mathbf{R}} \left({}^G\hat{\mathbf{p}}_{k+1} - {}^G\hat{\mathbf{p}}_k - {}^G\hat{\mathbf{v}}_k \Delta T\right) \rfloor - \mathbf{O}_\alpha \left(\tilde{q}_4 \mathbf{I} + \lfloor \tilde{\mathbf{q}} \rfloor\right) \tag{156}$$

$$\frac{\partial \mathbf{e}_p}{\partial \delta \mathbf{b}_{\omega_k}} = -\mathbf{J}_\alpha \tag{157}$$

$$\frac{\partial \mathbf{e}_p}{\partial^G \delta \mathbf{v}_k} = -{}^k_G\hat{\mathbf{R}} \Delta T \tag{158}$$

$$\frac{\partial \mathbf{e}_p}{\partial \delta \mathbf{b}_{a_k}} = -\mathbf{H}_\alpha \tag{159}$$

$$\frac{\partial \mathbf{e}_p}{\partial^G \delta \mathbf{p}_k} = -{}^k_G\hat{\mathbf{R}} \tag{160}$$

$$\frac{\partial \mathbf{e}_p}{\partial^G \delta \mathbf{p}_{k+1}} = {}^k_G\hat{\mathbf{R}} \tag{161}$$

where $[\tilde{\mathbf{q}}^\top \ \tilde{q}_4]^\top = {}^k_G\hat{\bar{q}} \otimes {}^k_G\bar{q}^{\star -1}$

Appendix A: Useful Identities

We provide some useful identities that are used in our derivations throughout the report. Given a constant angular velocity $\boldsymbol{\omega}$ between times t_1 and t_2, the rotation matrix between the two frames L_{t_1} and L_{t_2} is given by the matrix exponential:

$$\begin{aligned} {}^{L_{t_2}}_{L_{t_1}}\mathbf{R} &= \exp\left(-\lfloor \boldsymbol{\omega}(t_2 - t_1) \rfloor\right) \\ &= \mathbf{I}_{3\times 3} - \frac{\sin(|\boldsymbol{\omega}(t_2 - t_1)|)}{|\boldsymbol{\omega}|} \lfloor \boldsymbol{\omega} \rfloor + \frac{1 - \cos(|\boldsymbol{\omega}(t_2 - t_1)|)}{|\boldsymbol{\omega}|} \lfloor \boldsymbol{\omega} \rfloor^2 \end{aligned} \tag{162}$$

where the skew-symmetric is defined as:

$$\lfloor \boldsymbol{\omega} \rfloor = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{163}$$

The right Jacobian of $SO(3)$, $\mathbf{J}_r(\boldsymbol{\phi})$, is defined by (see [9]):

$$\mathbf{J}_r(\phi) = \mathbf{I}_{3\times 3} - \frac{1 - \cos(\parallel \boldsymbol{\phi} \parallel)}{\parallel \boldsymbol{\phi} \parallel^2} \lfloor \boldsymbol{\phi} \rfloor + \frac{\parallel \boldsymbol{\phi} \parallel - \sin(\parallel \boldsymbol{\phi} \parallel)}{\parallel \boldsymbol{\phi} \parallel^3} \lfloor \boldsymbol{\phi} \rfloor^2 \tag{164}$$

Given a small angle vector perturbation $\delta\boldsymbol{\phi}$, we can make the following approximation for the rotation matrix [7] :

$$\exp\left(\lfloor \boldsymbol{\phi} + \delta\boldsymbol{\psi} \rfloor\right) \simeq \exp\left(\lfloor \boldsymbol{\phi} \rfloor\right) \exp\left(\lfloor \mathbf{J}_r(\boldsymbol{\phi})\delta\boldsymbol{\phi} \rfloor\right) \tag{165}$$

This allows us to map a perturbation of the Lie algebra $so(3)$ to a perturbation on the group of $SO(3)$. The JPL (natural order) quaternion is used throughout the paper [2, 10], which parametrizes the rotation (162) as follows:

$${}^{L_{t_2}}_{L_{t_1}}\bar{q} = \begin{bmatrix} \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \sin\left(\frac{|\boldsymbol{\omega}(t_2 - t_1)|}{2}\right) \\ \cos\left(\frac{|\boldsymbol{\omega}(t_2 - t_1)|}{2}\right) \end{bmatrix} . \tag{166}$$

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